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ASSOCIATED SETS OF POINTS*

BY

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Introduction

Two sets of n points ordered with respect to each other, the one, P_n^k, in a linear space S_k, determined by the equations

$$(up_1) = 0, \quad (up_2) = 0, \quad \cdots, \quad (up_n) = 0,$$

and the other Q_n^{n-k-2}, in a linear space S_{n-k-2}, determined by the equations

$$(vq_1) = 0, \quad (vq_2) = 0, \quad \cdots, \quad (vq_n) = 0,$$

are called *associated sets* if the factors of proportionality in the coördinates of the points can be so chosen that an identity in u, v exists of the following form:

$$(1) \qquad (up_1)(vq_1) + (up_2)(vq_2) + \cdots + (up_n)(vq_n) \equiv 0.$$

This relation, obviously mutual, between the two sets is such that either set uniquely defines the other to within projective modifications. Some general properties of such sets have been given by the writer.‡

A characteristic algebraic property of two associated sets is that complementary determinants formed from the matrices of the coördinates of the two sets of points when taken so that (1) is satisfied are proportional. A characteristic geometric property is the following: On $k + 3$ of the points of P_n^k there is a unique rational norm curve N^k upon which the $k + 3$ points determine a set of $k + 3$ parameters; on the complementary set of $n - k - 3$ points of Q_n^{n-k-2} there is a pencil of linear spaces S_{n-k-3} whose members on the remaining $k + 3$ points determine a set of $k + 3$ parameters; these two sets of $k + 3$ parameters are projective.

Unless $k = n - k - 2$ the associated sets are in spaces of different dimension. Conventional methods of passing from one space to another are the process of *mapping* the space of lower dimension upon that of higher dimension, and the process of *projecting* from the space of higher dimension upon the

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‡ A. B. Coble, *Point sets and allied Cremona groups* (I), these Transactions, vol. 16 (1915), p. 155, in particular §§ 1, 2 and theorems (25), (26); also (II), vol. 17 (1916), p. 345, § 4 (16). These are cited as P. S. I or II.

one of lower dimension. Thus in the simple case of P_n^1, n points $x_0^{(i)}$, $x_1^{(i)}$ $(i = 1, \cdots, n)$ on a line, the line is mapped by means of the totality of binary $(n-3)$-ics in x, i.e., by $y_0 = (\alpha_0 x)^{n-3}, \cdots, y_{n-3} = (\alpha_{n-3}x)^{n-3}$, upon the points y of a rational norm curve N^{n-3} in S_{n-3} in such a way that P_n^1 is mapped upon its associated Q_n^{n-3}. On the other hand Q_n^{n-3} is projected from any S_{n-5} which is $(n-4)$-secant to N^{n-3} upon its associated P_n^1.

Two problems considered in this paper are: When $n - k - 2 \geqq k$ can the space S_k be mapped upon the space S_{n-k-2} so that the set P_n^k is mapped upon the set Q_n^{n-k-2}?; when $n - k - 2 > k$ can the set Q_n^{n-k-2} be projected upon the set P_n^k? For $k = 2$ the first problem is solved in § 1, the second in § 2. For $k = 3$ the first problem is solved in § 3. For the general set P_9^3 there appears to be no solution to the second problem and this probably would be true of further sets also.

In § 4 *particular* sets, i.e., those for which n, k have particular values, are considered. Each of these presents its own peculiarities. Also *special* sets, i.e., those which for given n, k satisfy in addition some projective conditions, receive some attention. Those conditions which are invariant under regular Cremona transformation of the set (cf. P. S. II, § 4) are especially emphasized. Their form in the two sets is often very diverse. Thus if P_9^3 is on a quadric with a node, then the associated Q_9^4 is on a rational quintic curve and conversely. In this section the discussion is carried through the values $n \leqq 10$.

The results obtained for the sets of nine and ten nodes of the rational sextic and of the symmetroid are useful in connection with the author's investigations of the modular functions of genus four attached to these figures.*

1. Mapping of P_n^2 upon its associated Q_n^{n-4}

The space S_2 is mapped upon the space S_{n-4} by means of a linear system Σ of ∞^{n-4} plane curves. The points of the plane are mapped upon a 2-way in S_{n-4} of order λ where λ is the number of variable intersections of two curves of Σ. The intersections of this 2-way by the linear S_{n-5}'s contained in S_{n-4} correspond in S_2 to the curves of Σ. We have therefore to find a system Σ so related to the set of points P_n^2 that the additional condition that three points of P_n^2 are on a line has as a consequence that there must exist a curve of Σ on the remaining $n-3$ points of P_n^2 and therefore also that the corresponding $n-3$ points of Q_n^{n-4} lie upon an S_{n-5} in S_{n-4}. This ensures the proportionality of complementary determinants in the matrices of the two point sets. Of course this requirement may not define the system Σ and we seek merely a simple system Σ with the required property.

* A part of this work appears in abstract in the Proceedings of the National Academy of Sciences, vol. 7 (1921), (I) p. 245; (II) p. 334. These are cited as Proc. I or II.

The cases where n is even and n is odd are slightly different and we begin with the mapping of a P^2_{2j+3} upon its associated Q^{2j-1}_{2j+3}. In S_2 pass through P^2_{2j+3} a proper curve C of order j with a $(j-2)$-fold point at a point r. Also pass through P^2_{2j+3} a proper curve D of order $(j+1)$ with a $(j-1)$-fold point at r which meets C in $(2j-5)$ points $s_1, s_2, \cdots, s_{2j-5}$. If L_1, L_2 are distinct lines on r, then D, CL, CL_2 cut out the same set S upon C, so that the set S lies in an I^{2j-5}_{j-3} on C. The choice of the points r, s thus depends upon $2j-5$ constants when P^2_{2j+3} is given. Let A, B respectively be arbitrary sets of $(j-2)$ and $(j-3)$ lines on r. Then in $AC + BD = 0$ we have a system of curves of order $(2j-2)$ with a $(2j-4)$-fold point at r, on P^2_{2j+3}, and on the points S. The parameters in A and B are essential. For if $AC + BD \equiv A'C + B'D$ then $(A - A')C \equiv (B' - B)D$, whence $A \equiv A'$ and $B \equiv B'$, since C and D are proper curves. Thus the system $AC + BD = 0$ contains ∞^{2j-4} curves. If three points of P^2_{2j+3} are on a line, a curve of the system can be passed through $(2j-4)$ further points of the line, which therefore will contain the line as a factor. The complementary factor will be a curve of the required system Σ of order $(2j-3)$ with a $(2j-4)$-fold point at r and on the set S, and this curve will pass through the complementary set of $2j$ points of P^2_{2j+3}. Hence the system Σ will map the set P^2_{2j+3} upon its associated set.

For the case n even, or a P^2_{2j+2}, we pass through P^2_{2j+2} two proper curves C, D of order j with a common $(j-2)$-fold point r which meet again in $(2j-6)$ points S. Here the choice of r, s depends upon $2j-6$ constants. Let A, B be arbitrary sets of $(j-3)$ lines on r. Then in $AC + BD = 0$ we have a linear system of dimension $(2j-5)$ of curves of order $(2j-3)$ with a $(2j-5)$-fold point at r, on P^2_{2j+2}, and on the set S. If three of the points of P^2_{2j+2} are on a line, one curve of the system contains this line as a factor, whence one curve of the required system Σ of order $(2j-4)$ with a $(2j-5)$-fold point at r and on S will pass through the complementary set of $(2j-1)$ points of P^2_{2j+2}. This system Σ therefore effects the required mapping. Hence

THEOREM 1. *The plane set of points P^2_n is mapped upon its associated Q^{n-4}_n by a linear system of curves of order $(n-6)$ with an $(n-7)$-fold point at r and on a set of $(n-8)$ points S in such a way that the plane is mapped upon the normal 2-way, M^{n-5}_2, of order $(n-5)$ in S_{n-4}. If n is even the points S are the further intersections of two proper curves of order $(n-2)/2$ with a common $(n-6)/2$-fold point at the arbitrarily chosen point r and on the given set P^2_n. If n is odd the points S are the further intersections of two proper curves of order $(n-3)/2$ and $(n-1)/2$ with respectively $(n-7)/2$- and $(n-5)/2$-fold points at r and on P^2_n. For given P^2_n the choice of the points r, S depends upon $(n-8)$ constants.*

The mapping described above becomes evanescent for $n = 6$ and $n = 7$. In the case of P_6^2 let a pencil of cubics on P_6^2 meet again in s_1, s_2, s_3. Then conics on S map P_6^2 upon its associated Q_6^2. For if three of the points of P_6^2 are on a line, the complementary three are on a conic with s_1, s_2, s_3 and therefore map into three points of Q_6^2 on a line. Hence

THEOREM 2. *Six corresponding point pairs of a quadratic transformation are associated P_6^2, Q_6^2 if P_6^2 and the singular triangle of the transformation are the base points of a pencil of cubics.*

In the case of P_7^2 we pass a pencil of cubics through P_7^2 to meet again in s_1, s_2. Then conics on s_1, s_2 map P_7^2 upon its associated Q_7^3 in S_3. In this mapping the plane becomes a quadric on Q_7^3 and the points on the line $\overline{s_1 s_2}$ become the directions on this quadric about the eighth base point of the net of quadrics on Q_7^3. Thus to the ∞^2 possible choices of the pair s_1, s_2 there correspond the set Q_7^3 and the ∞^2 quadrics on it.

We observe also that the cases $n = 8$, $n = 9$ are exceptional in that for P_8^2 r is the ninth base point of the pencil of cubics on P_8^2 and that for P_9^2 r is a point on the cubic determined by P_9^2. For further cases r may be taken in general position.

2. THE PROJECTION OF Q_{k+4}^k UPON ITS ASSOCIATED P_{k+4}^2

We now consider the set Q_{k+4}^k as given in S_k and ask for spaces L of dimension $k - 3$ such that under projection from L, the set Q will become its associated set in the plane. Two lemmas are needed.

LEMMA 1. *The S_{k-2} π determined by L and q_1 is a $(k - 1)$-secant space of the norm curve N_1^k on $q_2, \cdots, q_{k+4}$.*

For if τ is the parameter of the pencil of S_{k-1}'s on π and t the parameter on N_1^k the incidence condition of S_{k-1} τ and point t is a $(1, k)$ relation on τ, t which in general would have only $k + 1$ pairs τ, t in common with any $(1, 1)$ relation on τ, t. If this $(1, 1)$ relation is the projectivity mentioned in the introduction between the parameter τ of the line pencil on p_1 in S_2 and the parameter t of N_1^k, then it is satisfied by the $k + 3$ pairs t, τ determined by $q_2, \cdots, q_{k+4}$. Therefore the projectivity determines a $(1, 1)$ relation which is a factor of the $(1, k)$ relation. The complementary factor of degree $k - 1$ in t determines the points of N_1^k on π. Thus the $k + 4$ norm curves on the sets of $k + 3$ points q selected from Q_{k+4}^k are projected from L into $k + 4$ rational k-ics in the plane on the points of P_{k+4}^2 and with respectively a $(k - 1)$-fold point at each point of P_{k+4}^2. This remark is utilized in Theorem 5.

LEMMA 2. *Quadrics on $q_2, \cdots, q_{k+4}$ cut π in quadrics apolar to a unique quadric Q_π in π and L in π is the polar S_{k-3} of q_1 as to Q_π.*

For the $\binom{k}{2}$ linearly independent quadrics on N_1^k cut π in $\binom{k}{2}$ sections on

the $k-1$ points common to π and N_1^k, whence of these only $\binom{k}{2}-(k-1)$ are linearly independent in π. Therefore $k-1$ quadrics on N_1^k contain π and the $\binom{k+2}{2}-(k+3)$ quadrics in S_k on $q_2, \cdots, q_{k+4}$ cut π in at most $\binom{k-2}{2}-(k+3)-(k-1)=\binom{k}{2}-1$ linearly independent quadrics all of which are apolar to at least one quadric Q_π in π. Moreover the S_2 on three points of $q_2, \cdots, q_{k+4}$ and the S_{k-1} on the remaining k points meet π respectively in a point and S_{k-3} which are pole and polar as to Q_π and thereby Q_π is uniquely determined. For any S_{k-1} on S_2 together with the given S_{k-1} constitute a quadric on $q_2, \cdots, q_{k+4}$ and meet π in a pair of S_{k-3}'s apolar to Q_π. Finally, if three points of $q_2, \cdots, q_{k+4}$ are in an S_{k-1} with L and therefore project from L into three points of a line in S_2, then the remaining k points and q_1 must be in an S_{k-1} which meets π in an S_{k-3} on q_1. Hence the point, S_{k-3} of π mentioned above are such that when the point is on L then the S_{k-3} must be on q_1, which requires that q_1, L be pole and polar as to Q_π.

In order to put all the points of the set Q on the same footing we now prove

THEOREM 3. *Given Q_{k+4}^k in S_k there exist ∞^{k-3} spaces L of dimension $k-3$ such that all the quadrics on L and any $k+3$ of the points Q meet again at the remaining point of Q, or also such that all the quadrics on the points Q and $\binom{k-1}{2}-1$ points of L contain L. From any one of these spaces L the set Q_{k+4} is projected into its associated P_{k+4}^2.*

For there are ∞^{k-1} S_{k-2}'s which are $(k-1)$-secant spaces of N_1^k each with ∞^{k-2} points, so that on q_1 there are ∞^{k-3} such spaces π. In any such space π choose L to be the polar S_{k-3} of q_1 as to the quadric Q_π determined as in Lemma 2. Then all the quadrics of S_k on $q_2, \cdots, q_{k+4}$ which contain L cut π in another S_{k-3} on q_1 and L has the first property described in the theorem. That all the S_{k-3}'s L of the theorem are found among the $(k-1)$-secant spaces π of N_1^k on q_1 is proved as follows. If, as given, quadrics on $q_2, \cdots, q_{k+4}$ and L meet again in q_1, then the $\binom{k+2}{2}-\binom{k-1}{2}-(k+3)=2k-3$ linearly independent quadrics of this sort meet $\pi\,[L, q_1]$ in a linear system of S_{k-3}'s on p_1 of which only $k-2$ are linearly independent in π. Hence $k-1$ of the quadrics contain the S_{k-2} π and therefore meet in a N_1^k (necessarily on $q_2, \cdots, q_{k+4}$) which is $(k-1)$-secant to π. We observe that the configuration Q_{k+4}^k, L is the generalization of the set of eight base points of a net of quadrics as one of the points is enlarged in dimension. To prove the last statement in the theorem we note that if $q_2, \cdots, q_{k+2}$ are on an S_{k-1}, this S_{k-1} together with the S_{k-1} on L and q_{k+3}, q_{k+4} constitute a quadric which must contain q_1, whence in the projection p_1, p_{k+3}, p_{k+4} are on a line. Here the isolated position of p_1 is not material.

The above discussion suggests the following construction for the set in S_k when the set in the plane is given.

THEOREM 4. *Given the set P_{k+4}^2, let the parameter t of the line pencil on p_1 be*

introduced as a parameter on the linear system Σ_1 of ∞^{k-3} rational curves of order k with a $(k-1)$-fold point at p_1. Then $t_2, \cdots, t_{k+4}$ are the parameters of $p_2, \cdots, p_{k+4}$ on every curve of Σ_1 and the parameters of the multiple point p_1 determine a linear system of ∞^{k-3} binary $(k-1)$-ics all of which are apolar to a binary k-ic, γ_1^k. In S_k select a parameter system t on a norm curve N_1^k. Then the points of N_1^k with parameters $t_2, \cdots, t_{k+4}$ and the point of S_k determined by γ_1^k with reference to N_1^k constitute a set $q_2, \cdots, q_{k+4}, q_1$ associated with P_{k+4}^2.

This is indeed an immediate consequence of the fact that the curves of Σ_1 are the projections of N_1^k from the ∞^{k-3} spaces L. This same projection and the further fact that the choice of a single curve of the system Σ_1 is sufficient to determine the corresponding L lead to the following theorem, which is not readily apparent from the plane figure alone.

THEOREM 5. *The $k+4$ systems Σ_i of dimension $k-3$ of rational curves of order k with a $(k-1)$-fold point at p_i and simple points at the remaining points of P_{k+4}^2 are in one-to-one correspondence with each other.*

We shall see in § 4 that for Q_7^3 the $\infty^0 = 1$ space L is the point common to all of the ∞^2 elliptic quartics on Q_7^3; for Q_8^4 the ∞^1 spaces L are the common bisecants of all the ∞^1 elliptic quintics on Q_8^4; and for Q_9^5 the ∞^2 spaces L are the trisecant planes of the unique elliptic sextic on Q_9^5. For further sets no equally simple characterization of the spaces L has been obtained.

3. MAPPING OF P_n^3 UPON ITS ASSOCIATED Q_n^{n-5}

In order to map a set P_8^3 upon its associated Q_8^3 we need only to find a further set P_6^3 such that the set $P_{14}^3 = P_8^3 + P_6^3$ shall have the property that the linear system Σ of cubic surfaces on the 14 points shall have the dimension 6, i.e., that all the cubic surfaces on 13 of the points shall pass through the 14th. For then if 4 of the points of P_8^3 are in a plane π a cubic surface of the system Σ can be made to pass through 6 more points of π in general position and therefore to contain π as a factor. The remaining factor is a quadric on P_6^3 which contains the other four points of P_8^3. Hence the linear system of quadrics on P_6^3 will map P_8^3 upon its associated Q_8^3.

One symmetrical set of 14 points of such character may be obtained as follows. Given 6 points $r_1, \cdots, r_6$ of a plane, select a quartic curve with simple points at r and an octavic curve with triple points at r. These two curves meet elsewhere in 14 points. They are mapped from the plane by cubic curves on the points r into two space sextics of genus 3 with 14 common points. The two space sextics are on one cubic surface—the map of the plane —and only one since the two sextic curves could not lie at once on two cubic surfaces one of which is non-degenerate. Since each sextic curve is on 4 linearly independent cubic surfaces, there must be on their 14 common points $4+4-1=7$ linearly independent cubic surfaces and the set has the required property.

The trisecant locus of the one sextic—an octavic surface with the sextic as a triple curve—meets the other sextic in $8 \times 6 - 14 \times 3 = 6$ points, whence six trisecants of each curve are secants of the other and these two sets of trisecants are a double six of the unique cubic surface on both sextics—the double six of the mapping system. The rôles of the two plane curves are interchanged by the plane Cremona transformation of order 5 with double F-points at the six points r. We observe that the pair of space sextics is the complete intersection of a cubic and a quartic surface.

The number of absolute constants is 4 for the points r and 8 more for each of the plane curves, or 20 in all. Hence in space such a set of 14 points has $20 + 15 = 35$ projective constants. A space sextic of genus three has $15 + 9 = 24$ projective constants so that on a given sextic there are ∞^{11} such sets of 14 points which lie in a linear series I_{11}^{14}. From this there follows that at most 11 of the 14 points can be chosen at random in space. For such sets from P_8^3 to P_{11}^3 we have

THEOREM 6. *The three-dimensional sets P_8^3, P_9^3, P_{10}^3, and P_{11}^3 can be mapped upon their associated sets Q_8^3, Q_9^4, Q_{10}^5, and Q_{11}^6 by the linear system of quadrics on a supplemental set P_6^3, P_5^3, P_4^3, and P_3^3 respectively, which with the given set makes up the* 14 *points of intersection of two space sextics of genus three.*

The mapping system of this theorem is more general than is needful for the purpose. Consider for example the set P_8^3. It lies on a unique elliptic quartic E^4, the intersection of quadrics Q_1, Q_2. Let C be a cubic surface on P_8^3 which cuts E^4 in a residual set P_4. Let two other points in general position be a set P_2. The totality of cubic surfaces on the 12 points $P_8^3 + P_4$ is made up of $C + \pi Q_1 + \pi' Q_2$ where π, π' are arbitrary planes. In this system of ∞^8 surfaces there is a system of dimension 6 on $P_8^3 + P_4 + P_2$, whence quadrics on $P_4 + P_2$ map P_8^3 upon its associated set Q_8^3. This mapping is however a degenerate case of Theorem 6, since E^4 and a bisecant of E^4 from each point of P_2 make up a degenerate sextic of genus three.

The simplest transition from P_8^3 to Q_8^3 is obtained by taking P_8^3 on an E^4 with canonical parameter u (i.e., such that the coplanar condition is $u_1 + u_2 + u_3 + u_4 \equiv 0$ mod. ω_1, ω_2) for which the parameters of the points of P_8^3 are $u_1, \cdots, u_8$, where $\Sigma_1^8 u = \sigma$. If now we set $u_i + v_i = \sigma/4$ $(i = 1, \cdots, 8)$ then $v_1 + \cdots + v_4 \equiv \sigma - (u_1 + \cdots + u_4) \equiv u_5 + \cdots + u_8$. Hence the four points v are on a plane if the complementary four points u are on a plane, or the set v is associated to the set u. The lines joining u_i, v_i are generators of a regulus on E^4. For given P_8^3 the $\sigma/4$ has 16 determinations, whence

THEOREM 7. *For a given set P_8^3 there are* 16 *reguli on the E^4 through P_8^3 such that the generators of a regulus on the points of P_8^3 meet the E^4 again in the points of an associated Q_8^3.*

Again let the set P_9^3 be on a quadric with generators t, τ and let $(a\tau)^2(\alpha t)^3$

$= 0$ and $(b\tau)^3(\beta t)^2 = 0$ be two quintics of genus two of different kinds on P_9^3 and Q. These quintics meet in four other points P_4 on Q. Let P_1 be a point in general position. Then if C_1, C_1' are cubic surfaces on the first quintic, C_2, C_2' cubic surfaces on the second quintic, and π is an arbitrary plane we have in $\lambda_1 C_1 + \lambda_2 C_1' + \lambda_3 C_2 + \lambda_4 C_2' + \pi Q$ a system of ∞^7 cubic surfaces on P_9^3 and P_4. Hence there will be a system of dimension 6 on P_9^3, P_4, P_1, or the system of quadrics on $P_4 + P_1$ will map P_9^3 upon Q_9^4. This again is a special case of Theorem 6 since a bisecant to the one quintic from P_1 makes up with the quintic a degenerate sextic of genus three and the two sextics thus made up have 14 common points. We shall however find in § 4 a different mode of transition from P_9^3 to Q_9^4 which exhibits more effectively their mutual relations.

There appears to be no point in S_4 from which a general set Q_9^4 can be projected into its associated set. If Q_9^4 is on an elliptic quintic E^5 (two conditions) a quadric on Q_9^4 will cut E^5 in a tenth point from which the desired projection can be made (§ 4, Theorem 11). However, no general sets except planar sets have been found which are the projections of their associated sets. On the other hand no proof of the impossibility of such a projection has been found.

We complete the mapping of sets P_n^3 upon their associated sets by means of an apparatus derived from the elliptic curves. Let E_k^m be an elliptic curve of order $m > k$ in an S_k. It is the projection of the normal E_{m-1}^m from an S_{m-k-2}. The E_{m-1}^m has one absolute constant and the S_{m-k-2} in S_{m-1} has $(m-k-1)(k+1)$ further constants, so that the projection has $(m-k-1)(k+1)+1$ absolute constants. This number added to the $(k+1)^2 - 1$ constants of a projectivity in S_k furnishes $m(k+1)$. Hence the elliptic m-ic in S_k, E_k^m, has $m(k+1)$ constants and can be passed through $[m(k+1)/(k-1)]$ points in S_k, where the bracket indicates the largest integer equal to or less than the number within it.

Since r-ic spreads cut the E_k^m in an I_{mr-1}^{mr}, an r-ic spread on mr general points of E_k^m contains it completely. Hence there are $\infty^{\binom{r+k}{k}-mr-1}$ r-ic spreads on E_k^m and there are $\infty^{\binom{r+k}{k}-mr}$ r-ic spreads on the mr points cut out on E_k^m by a definite r-ic spread.

Beginning then with a set P_{2j}^3 we can pass an E_3^j through its points. Let an r-ic surface on P_{2j}^3 meet E_3^j in $j(r-2)$ further points $P_{j(r-2)}$. Then there are $\infty^{\binom{r+3}{3}-jr}$ r-ic surfaces on $P_{2j}^3 + P_{j(r-2)}$. If we suppose that these surfaces are subject to $\alpha \geqq 0$ further linear conditions, say to pass through a set of points P_α, we have a linear system of $\infty^{\binom{r+3}{3}-jr-\alpha}$ r-ic surfaces on the base $P_{2j}^3 + P_{j(r-2)} + P_\alpha$. If 4 points of P_{2j}^3 are on a plane and if $\binom{r+3}{3} - jr - \alpha = \binom{r+2}{2} - 4$, then an r-ic surface of the linear system can be determined which contains this plane as a factor leaving an $(r-1)$-ic surface on $P_{j(r-2)} + P_\alpha$

which passes through the remaining $2j-4$ points of P^3_{2j}. This condition becomes

(2) $$\binom{r+2}{3} - jr + 4 = \alpha .$$

Since $\alpha \geqq 0$, then, for given j, r is defined by the inequality

(3) $$\binom{r+2}{3} + 4 \geqq jr .$$

The modification for an odd set P^3_{2j-1} is readily made and we state at once

THEOREM 8. *Through a given set $P^3_{2j}\{P^3_{2j-1}\}$ pass an E^j_3 and cut it by an r-ic surface on $P^3_{2j}\{P^3_{2j-1}\}$ which meets E^j_3 again in a set $P_{j(r-2)}\{P_{j(r-2)+1}\}$ where r is the smallest integer defined by* (3). *The linear system of surfaces of order $r-1$ on this residual set and on a further general set P_α, where α is defined by* (2), *maps S_3 upon a 3-way in $S_{2j-5}\{S_{2j-6}\}$ in such a way that the set $P^3_{2j}\{P^3_{2j-1}\}$ is mapped upon its associated $Q^{2j-5}_{2j}\{Q^{2j-6}_{2j-1}\}$.*

For the sets P^3_9 and P^3_{10} the numbers j, r, α are $5, 4, 4$; for P^3_{11} and P^3_{12}, $6, 4, 0$; for P^3_{13} and P^3_{14}, $7, 5, 4$; etc.

4. Particular and special sets of points

It is the aim in the present section to consider in more detail the relation of particular sets P^k_n for values of n from 8 to 10 to their associated sets both for cases when the n points of the set are in general position and for cases when they are subject to certain conditions. A question naturally arises as to what types of conditions would be most interesting and as to what types of configurations connected with the associated sets would best exhibit the relations sought. In answer to this inquiry we recall the noteworthy theorem in regard to associated sets (P. S., II (16), p. 361), which states that if P^k_n and P'^k_n are congruent under regular Cremona transformation in S_k their associated sets Q^{n-k-2}_n and Q'^{n-k-2}_n are also congruent under regular Cremona transformation in S_{n-k-2}. More specifically, if P^k_n is congruent to P'^k_n under the Cremona involution $x'_i = 1/x_i$ $(i = 1, \cdots, k+1)$ with its $k+1$ F-points at points of P^k_n, then Q^{n-k-2}_n is congruent to Q'^{n-k-2}_n under the involution $x'_i = 1/x_i$ $(i = 1, \cdots, n-k-1)$ with its $n-k-1$ F-points at the complementary $n-k-1$ points of Q^{n-k-2}_n. The regular Cremona group is generated by this one Cremona involution and projectivities.

We shall seek therefore to express the desired relations in terms of such loci or in terms of such properties of these loci as are invariant under regular Cremona transformation. Thus a rational curve, or an elliptic curve, of order $k+1$ on the points of P^k_n is transformed by regular transformation into a curve of the same order on the points of the congruent set. The same is true of multiples of such curves, i.e., curves of orders $l(k+1)$ with l-fold points at the points of P^k_n, if such curves exist. This property of invariance is shared by a certain type of surface—the rational M^r_2 in S_{r+1}. We shall first derive some facts concerning this surface for later use.

If $r = 2l + 1$ $[2l]$ the system of rational plane curves of order $l + 1$ on the base O^l $[O^l, \sigma]$ has the dimension $r + 1$ and maps the plane upon a 2-way of order r, M_2^r, in S_{r+1}. Each of these surfaces is the projection of the one of next higher order from one of its points. This is evidently the case in passing from the base O^l to the base O^l, σ. But also the base O^l, σ, σ' can be reduced by quadratic transformation to the base O^{l-1}. Thus the series of surfaces M_2^r constitute the progenitors of the quadric M_2^2 in S_3. Lines on the point O map into the ∞^1 "generators" of the surface.

In case r is odd directions at O map into a unique "directrix," a rational norm curve of order l; while the lines of the plane map into ∞^2 "directors," rational norm curves of order $l + 1$. Since S_r's on the directrix are mapped by sets of $l + 1$ lines on O, and S_r's on a given director by sets of l lines on O and a given line, the directrix and a director are in skew S_l, S_{l+1}, and the generators are lines joining corresponding points of these two rational curves. Included, however, among the ∞^2 directors are the ∞^1 which consist of the fixed directrix and a variable generator.

In case r is even there are ∞^1 directrices, the maps of lines on σ, which are rational norm curves of order l. Included in this system is one curve which is the map of directions at O. As before the ∞^1 generators are the maps of lines on O but this system includes the one line which is the map of directions at σ. The line $\overline{O\sigma}$ is mapped into directions on the surface about the point where the generator σ meets the directrix O. If π, ρ are two lines on σ the mapping system can be expressed in the form $\pi\Sigma_1 + \rho\Sigma_2$ where Σ_1, Σ_2 each is the system of l lines on O. Hence any two of the directrices lie in skew S_l's and the generators are lines joining corresponding points on the two.

In either case by estimating the number of constants involved in the choice of the skew spaces; in the choice of the rational curve in each; in the projectivity between the two curves set up by the generators; and by allowing for the freedom in the choice of the skew spaces for given surface, we find that the number of projective constants of the M_2^r is $(r + 2)^2 - 7$, whence the M_2^r admits a 6-parameter collineation group. This group for r odd is the map of the 6-parameter collineation group of the plane with fixed point O; for r even it is the map of the 6-parameter quadratic group with fixed F-points at O, σ.

Since it is $r - 1$ conditions that an M_2^r in S_{r+1} be on a point, we see that there are ∞^2 M_2^r's on $r + 5$ points in general position. Thus on 8 points in S_4 there are ∞^2 M_2^3's, or on 9 points a finite number; on 9 points in S_5 there are ∞^2 M_2^4's which fill up a spread, whence for 10 points in S_5 there is a single condition invariant under regular Cremona transformation which expresses that the 10 points lie on an M_2^4.

The system of plane rational curves of order l on the base O^{l-1} $[O^{l-1}, \sigma]$

has the dimension $r - 1$. Let C_i $(i = 1, \cdots, r)$ be linearly independent in this system and let π, ρ be two lines on O. If then we set $m_i = \pi C_i$, $n_i = \rho C_i$, where m_i, n_i are the linear forms in S_{r+1} which cut M_2^r in the maps of the given plane curves, we find that the equation of M_2^r is

$$\left\| \begin{matrix} m_1 & m_2 & \cdots & m_r \\ n_1 & n_2 & \cdots & n_r \end{matrix} \right\| = 0. \tag{4}$$

Conversely a manifold in S_{r+1} defined by such a matrix is in general an M_2^r mapped as above.

In the case $r = 2l$ a parametric equation of M_2^r is

$$x_0 = (\alpha_0 t)(a_0 \tau)^l, \quad x_1 = (\alpha_1 t)(a_1 \tau)^l, \quad \cdots, \quad x_{r+1} = (\alpha_{r+1} t)(a_{r+1} \tau)^l. \tag{5}$$

For given τ we have one of the ∞^1 generators; for given t one of the ∞^1 directrices. In the case $r = 2l + 1$ the parametric equation is

$$\begin{gathered} x_0 = (\alpha_0 t)(a_0 \tau)^{l+1}, \qquad x_1 = (\alpha_1 t)(a_1 \tau)^{l+1}, \qquad \cdots, \\ x_{r+1} = (\alpha_{r+1} t)(a_{r+1} \tau)^{l+1}, \end{gathered} \tag{6}$$

where

$$(\alpha_i t')(a_i \tau')^{l+1} = 0 \qquad (i = 0, \cdots, r+1).$$

This is in fact the projection of (5) for $r = 2l + 2$ from a point t', τ' upon it.

Special cases of these rational surfaces occur. Thus cubic curves on the base O^2, o, o' map the plane upon an M_2^3 in S_4. This mapping system can be reduced to conics on the base O by quadratic transformation with F-points at O, o, o' unless o, o' coincide with O in two distinct directions. Thus cubics with node at O and fixed nodal tangents determine an M_2^3 in S_4 which is more properly the projection of an M_2^5 in S_6 from two points on its directrix conic. This special M_2^3 is obtained in S_4 by joining a point directrix to a cubic curve director. Unless expressly mentioned special M_2^r's of such types will not be considered.

We shall now prove

THEOREM 9. *An M_2^r in S_{r+1} is transformed by the Cremona involution $x'_i = 1/x_i$ $(i = 1, \cdots, r+2)$ with $r + 2$ F-points on the M_2^r into an $M_2^{r'}$.*

The space x' is mapped in the involution upon the space x by the system of spreads of order $r + 1$ with r-fold points at the F-points which are the maps from the plane of the points $p_1, \cdots, p_{r+2}$. Then, for $r = 2l + 1$, the transform of M_2^r is mapped from the plane by curves of order $2(l+1)^2$ with a $2l(l+1)$-fold point at O and $(2l+1)$-fold points at $p_1, \cdots, p_{2l+3}$; for $r = 2l$, by curves of order $(l+1)(2l+1)$ with an $l(2l+1)$-fold point at O, a $(2l+1)$-fold point at σ; and $2l$-fold points at $p_1, \cdots, p_{2l+2}$. We have merely to show that the two latter mapping systems can be transformed by ternary Cremona transformation into systems of order $l + 1$ on the bases O^l or O^l, σ respectively. For odd r this transformation is effected by using first the Jonquière transformation J^{l+1} of order $l + 1$ with l-fold point (*center*)

at O and simple F-points at $p_1, \cdots, p_{2l}$, then a quadratic transformation with F-points at $p_{2l+1}, p_{2l+2}, p_{2l+3}$, and finally the transformation J^{l+1} again. For even r we use first a quadratic transformation with F-points at O, σ, p_1, then a J^l with center at O and simple F-points at $p_2, \cdots, p_{2l-1}$, then the quadratic transformation with F-points at $p_{2l}, p_{2l+1}, p_{2l+2}$, and finally J^l again. It is easily verified that these transformations effect the required change in the mapping system and the proof is complete.

The three theorems which follow relate to special sets of points when, for given k, n is sufficiently large.

THEOREM 10. *If P_n^k is on a rational norm curve N^k in S_k, then its associated Q_n^{n-k-2} is on a rational norm curve N^{n-k-2} in S_{n-k-2}. The n parameters of the two sets on their respective norm curves are projective. If $(n-k-2)-k = l+1 > 0$, the set Q is projected upon the set P from any one of the ∞^{l+1} spaces L_l which are $(l+1)$-secant to N^{n-k-2}.*

THEOREM 11. *If P_n^k is on an elliptic norm curve E^{k+1} in S_k, then its associated Q_n^{n-k-2} is on an elliptic norm curve E^{n-k-1} in S_{n-k-2}. If $(n-k-2)-k = l+1 > 0$, the set Q is projected upon the set P from any one of the ∞^l spaces L_l which are $(l+1)$-secant to E^{n-k-1} at the $l+1$ points cut out by a quadric on Q.*

THEOREM 12. *If P_n^k is on a rational norm surface M_2^{k-1} in S_k, then its associated Q_n^{n-k-2} is on a rational norm surface N_2^{n-k-3} in S_{n-k-2}. Then parameters of the two sets of generators on the points are projective.*

In Theorem 10 let the norm curves in S_k and S_{n-k-2} have the respective parametric equations

$$x_0 = 1, \quad x_1 = t, \quad \cdots, \quad x_k = t^k;$$

$$x_0 = 1, \quad x_1 = t, \quad \cdots, \quad x_{n-k-2} = t^{n-k-2};$$

and let the sets P_n, Q_n be determined on these curves by the parameters $t_1, \cdots, t_n$. If $\lambda_1, \cdots, \lambda_n$ are determined by the $n-1$ equations

$$\lambda_1 t_1^i + \lambda_2 t_2^i + \cdots + \lambda_n t_n^i = 0 \qquad (i = 0, 1, \cdots, n-2),$$

then the points of the one set, affected respectively by factors of proportionality $\lambda_1, \cdots, \lambda_n$, satisfy with the points of the other the bilinear relations requisite for association. We observe that here P_n^2 is obtained by projection of Q_n^{n-4} from ∞^{n-6} spaces L_l rather than ∞^{n-7} spaces as in the general case of Theorem 3.

In Theorem 11 let the canonical parameters of P_n^k on E^{k+1} be $u_1, \cdots, u_n$ where $u_1 + \cdots + u_n + b \equiv 0$. Choose then a mapping system on a base B such that the members meet E^{k+1} in $n-k-1$ variable points and also in a certain number of fixed points whose parameters sum up to b. Then, if $k+1$ points of P_n^k are on an S_{k-1}, $u_1 + \cdots + u_{k+1} \equiv 0$ and $u_{k+2} + \cdots + u_n + b \equiv 0$, whence the complementary $n-k-1$ points of P_n^k are on a member of the mapping system or the $n-k-1$ points of Q_n^{n-k-2}, mapped

from P_n^k, are on an S_{n-k-3}. Thus E^{k+1} is mapped upon E^{n-k-1} and P_n^k is mapped upon its associated Q_n^{n-k-2}. In this way we find upon each of the associated sets P_7^2, Q_7^3, ∞^2 elliptic norm curves, upon each of the associated sets P_8^2, Q_8^4, ∞^1 elliptic norm curves, and upon each of the associated sets P_9^2, Q_9^5, a unique elliptic norm curve.

For Theorem 12 we give the details of the proof only for the case $k = 2l + 1$. Then M_2^{k-1} is the map of the plane by curves of order $l + 1$ on the base O^l, σ and P_n^k is the map of a set π_n^2 in the plane. If $n = 2m + 1$ then O is the center and σ, π_n^2 the simple F-points of a J^{m+2} whose inverse center and F-points are O', σ', $\pi_n'^2$. Curves of order $m - l - 1$ on the base O'^{m-l-2} map the plane on an M_2^{n-k-3} and map $\pi_n'^2$ upon a set Q_n^{n-k-2} which is associated to P_n^k. For if $k + 1$ points $p_1, \cdots, p_{2l+2}$ of P_n^k are on an S_{k-1} there is a curve of order $l + 1$ with l-fold point at O and simple points at σ, $\pi_1, \cdots, \pi_{2l+2}$. This curve is transformed by J^{m+2} into a curve of order $m - l - 1$ with $(m - l - 2)$-fold point at O' and simple points at $\pi'_{2l+3}, \cdots, \pi'_{2m+1}$. Hence the points $q_{2l+3}, \cdots, q_n$ are on an S_{n-k-3} in S_{n-k-2}. If, however, n is even we take σ, σ' to be a pair of ordinary corresponding points for a $J^{1+\frac{n}{2}}$.

It should be observed however that an M_2^2 in S_3, an ordinary quadric, counts in two ways as a ruled normal surface. It is mapped from the plane by conics on O, σ and as the points are interchanged in the above proof two normal surfaces in S_{n-k-2} are obtained. Hence

THEOREM 13. *If P_n^3 is on a quadric surface which is not a cone, its associated Q_n^{n-5} is on two normal M_2^{n-6}'s in S_{n-5}.*

A simple statement of the relations among the F-points of a Jonquière transformation can be given in terms of associated sets.

THEOREM 14. *Given the Jonquière transformation J^{n+1} with center at p and simple F-points at P_{2n}^2, then curves of order $n - 2$ with an $(n - 3)$-fold point at p map the plane upon an M_2^{2n-5} in S_{2n-4} and map the set P_{2n}^2 upon a set R_{2n}^{2n-4} which is associated to the set Q_{2n}^2 of simple F-points of the inverse transformation.*

The proof of this is immediate by the foregoing methods.

We now proceed to particular sets beginning with P_8^2, Q_8^4. The ∞^1 elliptic quintics, E^5's, on Q_8^4 are obtained by the mapping of P_8^2 on Q_8^4 by conics on the 9th base point p_9 of the pencil of cubics on P_8^2. This pencil becomes a pencil of E^5's on an M_2^3 on Q_8^4 and the generators of M_2^3, which arise from the lines of the plane on p_9, are bisecants of all these E^5's. However, each of the ∞^1 E^5's on Q_8^4 has ∞^1 M_2^3's on it, whose generators on points v_1, v_2 satisfy the involution $v_1 + v_2 \equiv k$.* That particular M_2^3 common to all the E^5's is determined by the involution cut out on any E^5 by quadrics on Q_8^4. For if, in the plane, $u_1 + u_2 + \cdots + u_8 + u_9 \equiv 0$, $v_1 + \cdots + v_5 + u_9 \equiv 0$, $w_1 + w_2 + u_9 \equiv 0$ represent the sections of a cubic of the pencil by respectively a cubic

*Segre, Mathematische Annalen, vol. 27 (1886).

of the pencil, a mapping conic, and a line on p_9, then, on writing the second relation in the form $(v_1 + \frac{1}{5}u_9) + \cdots + (v_5 + \frac{1}{5}u_9) \equiv 0$ in order to introduce the canonical parameter $v' = v + \frac{1}{5}u_9$ on the mapped E^5, we have for Q_8^4 and the meets of a generator of the unique M_2^3 the relations

$$(u_1 + \tfrac{1}{5}u_9) + \cdots + (u_8 + \tfrac{1}{5}u_9) \equiv \tfrac{8}{5}u_9, \quad (w_1 + \tfrac{1}{5}u_9) + (w_2 + \tfrac{1}{5}u_9) \equiv -\tfrac{8}{5}u_9,$$

whence on E^5 $v_1' + \cdots + v_8' + w_1' + w_2' \equiv 0$ and the ten points are a quadric section.

We may relate Q_8^4 and any one of the ∞^2 M_2^3's on it to P_8^2 in the plane as follows. Let P_8^2, R_8^2 be F-points of a J^5 with centers at p, r where p is any one of the ∞^2 points of the plane. Then if p_1, p_2, p_3 are on a line, the points $r_4, \cdots, r_8, r$ are on a conic. Hence conics on r map the plane upon an M_2^3 in S_4 in such a way that R_8^2 is mapped upon the set Q_8^4 associated to P_8^2.

In addition to the ∞^1 E^5's on Q_8^4 there are ∞^2 rational quintics R^5 on Q_8^4. These are in one-to-one correspondence with the ∞^2 M_2^3's on Q_8^4. For, given an M_2^3 on Q_8^4, of the 7 linearly independent quadrics on Q_8^4 three are on M_2^3 (the three determinants of the matrix (4)) and of the remaining four one is on the directrix of M_2^3 and cuts M_2^3 in a residual R^5 trisecant to the directrix and unisecant to the generators. Conversely, given an R^5 on Q_8^4 it has a unique trisecant (with parameters determined by the canonizant of the binary quintic apolar to all S_3 sections) whose points are in 1–1 correspondence with the points of the curve (the correspondence being determined by making the three points common to the curve and trisecant self-corresponding) and the lines joining corresponding points are generators of an M_2^3 on Q_8^4. The question then arises as to the nature of the spread which is the locus of the ∞^2 R^5's on Q_9^4 or the nature of the condition that a Q_9^4 be on an R^5, and as to the corresponding condition on the associated P_9^3. The two theorems which follow answer these questions.

Theorem 15. *There are two M_2^3's on a given Q_9^4 which are covariantly related to the set under regular Cremona transformation. They are isolated by the same irrationality as separates the two reguli on the unique quadric on the associated set P_9^3. The parameters of the 9 generators of one of the M_2^3's on Q_9^4 are projective to those of the 9 generators of one of the reguli on P_9^3. If the set Q_9^4 lies on an R^5 (a single condition) then it lies on but one M_2^3 and its associated P_9^3 lies on a quadric cone.*

Two M_2^3's in S_4 meet in a set Q_9^4. That on Q_9^4 there are two M_2^3's is proved by Theorem 13. That there are only two is proved as follows. The ∞^2 M_2^3's on Q_8^4 are loci of ∞^1 bisecants of the ∞^1 E^5's on Q_8^4. One of these M_2^3's, say m_2^3, is a locus of bisecants of each of the E^5's; the others are each a bisecant locus of only one E^5. If then an M_2^3 is on a 9th point q_9 there is a bisecant of an E^5 on q_9; if two M_2^3's are on q_9 their plane cuts m_2^3 in the 4 meets of two

bisecants with their respective E^5's. Hence this plane cuts m_2^3 in one of the director conics on it. A third bisecant on Q_9 would have to be in this plane else there would be two director conics with two intersections, whereas such conics have only one. A director conic meets each of the ∞^1 E^5's in three points and on this conic there is an involution of triads whose joining triangles envelop another conic. Hence on the point q_9 in the plane of this conic there are just two lines of this envelope each belonging to one of the two M_2^3's on Q_9^4.

If Q_9^4 is on an R^5 which must lie on one of the two M_2^3's on Q_9^4 and must cut its directrix in three points and each generator in one point, then in the notation of the proof of Theorem 12 the R^5 must be the map of a rational plane quartic with triple point at O' and on $\pi_8', \cdots, \pi_8'$ as well as on σ'. But then σ must coincide in some direction with O, and the quadric on P_9^3 mapped by conics on O, σ is a quadric cone.

An E^5 in S_4 is projected from a line into an elliptic plane quintic with five nodes and from a line which meets E^5 into an elliptic plane quartic with two nodes, whence the bisecant locus of E^5 is a quintic spread on which E^5 is a triple curve. The ∞^1 E^5's on a given Q_8^4 can be put into 1, 1 correspondence with a pencil of plane cubics and therefore can be named rationally in terms of a parameter λ. Through a point there pass two bisecants belonging to two of these E^5's, whence the aggregate of these bisecant spreads of the ∞^1 E^5's constitute a quadratic system. The two bisecants isolate the two M_2^3's on Q_8^4 and the given point, whence if they coincide the two M_2^3's coincide and the given point and Q_8^4 are on an R^5. Hence

THEOREM 16. *If $\lambda^2 B_0 + 2\lambda B_1 + B_2 = 0$ is the quadratic system of bisecant spreads of the ∞^1 E^5's on Q_8^4, the spread $B_1^2 - B_0 B_2 = 0$ (a spread of order 10 with 6-fold points at Q_8^4 and a double M_2^3 consisting of the ∞^1 E^5's) is the locus of the ∞^2 rational quintics on Q_8^4, or the locus of points through which there can be drawn but one line bisecant to an E^5 on Q_8^4, or through which there can be passed but one M_2^3 on Q_8^4. Its equation may be obtained by replacing in the condition that a quadric on P_9^3 be nodal (a condition of degree 8 in the coördinates of each point of P_9^3 whose terms consist of products of 18 determinants $|p_i\, p_j\, p_k\, p_l|$) each determinant $|p_{i_1} p_{i_2} p_{i_3} p_{i_4}|$ by the complementary determinant $|q_{i_5}q_{i_6}q_{i_7}q_{i_8}q_{i_9}|$ formed for Q_9^4 and allowing the 9th point to vary.*

Here then we have an instance of the actual determination of a covariant of Q_8^4 or an invariant of Q_9^4 under the infinite group of regular Cremona transformations attached to the set.

We complete the discussion of sets of 9 points with the Q_9^5 associated with the set P_9^2. In S_5 the elliptic norm sextic E^6 has one absolute and 36 projective constants; the rational sextic R^6 has three absolute and 38 projective constants; and the M_2^4 has 29 projective constants; whence on Q_9^5 there is a finite number of E^6's, ∞^2 R^6's, and ∞^2 M_2^4's. There is, however, in S_5 a new

type of rational 2-way of order 4, the Veronese surface V_2^4, which shares with M_2^4 the property that its projection from one of its points is an M_2^3. The V_2^4 is the map of the plane by the linear system of all conics in the plane. It contains ∞^2 conics, the maps of lines of the plane, and the locus of the ∞^2 planes of these conics is a V_4^3 upon which V_2^4 is a double manifold. Analytically V_4^3 is obtained by setting a 3-row symmetric determinant of linear forms equal to zero and V_2^4 is the locus for which the six first minors vanish. The V_2^4 is unaltered by an 8-parameter collineation group, the map of the ternary group, whence it has $35 - 8 = 27$ projective constants. We should expect, therefore, to find on Q_9^5 a finite number of V_2^4's. The surface V_2^4 shares with M_2^4 also the property expressed by

THEOREM 17. *The Veronese surface V_2^4 is transformed into a Veronese surface $V_2'^4$ by a regular Cremona transformation whose F-points are on V_2^4. If the regular transformation in S_5 is $y_i = 1/x_i$ $(i = 0, \cdots, 5)$ the two V_2^4's are mapped by conics from planes which are in correspondence under the ternary quintic transformation with 6 double F-points. The V_4^3 with double V_2^4 is transformed into the $V_4'^3$ with double $V_2'^4$.*

Indeed the given involution maps the $S_4(y)$'s upon a system of quintic spreads with 4-fold points at the 6 F-points on V_2^4. This is the map of a ternary system of 10-ics with 4-fold points at 6 points, which can be transformed by the ternary transformation mentioned into a system of conics. The same involution transforms a cubic spread with nodes at the 6 F-points into a similar spread, whence V_4^3 on V_2^4 passes into $V_4'^3$ on $V_2'^4$.

Upon V_2^4 there is a linear system of ∞^9 E^6's, the maps of cubic curves in the plane. Conversely an E^6 on V_2^4 is cut out by a quadric which meets V_2^4 in a residual conic, whence the corresponding quartic in the plane breaks up into a line and a cubic. Therefore there are no other E^6's on V_2^4. The conics on V_2^4 are trisecant to the E^6's on V_2^4. A canonical elliptic parameter on the plane cubic is mapped into a canonical parameter on E^6 whence the planes of V_4^3 are those which meet E^6 in three points for which $u_1 + u_2 + u_3 \equiv 0$. Obviously any two of these planes lie in an S_4 and meet in a point. But the same thing is true of the three other involutions for which $u_1 + u_2 + u_3 \equiv \omega/2$. Hence on E^6 there are 4 V_2^4's or also there are 4 V_4^3's which contain E^6 doubled. Such a V_4^3 must contain every bisecant of E^6. The locus of bisecants, B_3^9, of E^6 is a 3-way of order 9 which has E^6 as a 4-fold curve, since from a plane E^6 is projected into a plane sextic with 9 nodes, and from a plane which meets E^6 the E^6 is projected into a plane quintic with 5 nodes. Hence the bisecant locus is the complete intersection of two of the four V_4^3's and the four lie in a pencil. A member of this pencil other than a V_4^3 also contains B_3^9. Given then a trisecant plane for which $u_1 + u_2 + u_3 \equiv k$, the above pencil of W_4^3's contains the three bisecants in the plane, whence one member, say W_4^3, contains

the plane. Since any plane for which $v_1 + v_2 + v_3 \equiv -k$ meets the above plane in a point, W_4^3 must meet this latter plane in its bisecants and an outside point and therefore must contain it. Hence W_4^3 is the locus of the ∞^2 trisecant planes $v_1 + v_2 + v_3 \equiv -k$ or also of the ∞^2 trisecant planes $u_1 + u_2 + u_3 \equiv k$. For each of the 4 V_4^3's in the pencil of W_4^3's the two systems of generating planes coincide into a single system, since $k \equiv -k$ when $k \equiv \omega/2$. Hence

THEOREM 18. *An E^6 is contained on 4 V_2^4's whose V_4^3's are in the pencil of spreads W_4^3 on the bisecant locus B_3^9 of E^6 for which E^6 is a 4-fold curve. A particular W_4^3 of the pencil with double E^6 has the two systems of ∞^2 generating trisecant planes for which $u_1 + u_2 + u_3 \equiv -k$, k which coincide for the 4 V_4^3's. Under regular Cremona transformation with F-points on E^6 the properties of this pencil are invariant.*

That there are on E^6 four V_2^4's may be seen by the use of an elementary theorem. Isolate one of the V_2^4's as the map of a plane. The V_4^3's of the other three V_2^4's cut the isolated one in E^6 doubled, whence in the plane we have the square of a cubic expressed in three ways as a symmetric 3-row determinant whose elements are conics. But we know that a cubic can be expressed in three ways as a symmetric 3-row determinant of linear forms, since it is the hessian of three cubics and the square of a symmetric determinant is symmetric. Moreover we know that the relation of the hessian to the three cubics involves the three half periods.

THEOREM 19. *On a general set Q_9^5 there is a unique E^6 and four V_2^4's.*

We see at once that an E^6 and an E'^6 on Q_9^5 could not have different absolute invariants. For an E^6 on Q_9^5 is projected from a properly chosen trisecant plane into an E^3 on the associated P_9^2, and E'^6 into an E'^3 on P_9^2, whence, since E^3 and E'^3 cannot coincide, the set P_9^2 is the special set of 9 base points of a pencil of E^3's and Q_9^5 is also a special set. If, however, there were an E^6 and an E'^6 on Q_9^5, then on projecting from q_9 we should have in S_4 an E^5 and E'^5 on R_8^4, members of a pencil on an M_2^3 in S_4. Hence in S_5 there are ∞^1 elliptic quintic 2-way cones with vertex at q_9 and on $q_1, \cdots, q_8$, and with no other points common to any two. A quadric on Q_9^5 and four generators of any one of these cones meets the cone in an E^6 on Q_9^5, whence there is a pencil of such E^6's on Q_9^5 with all values of the absolute invariant and again Q_9^5 is the special set above. This unique E^6 and therefore Q_9^5 also carries four V_2^4's. There are no V_2^4's on Q_9^5 which are not also on E^6, else there would be on such a V_2^4 an E'^6 on Q_9^5.

THEOREM 20. *If P_9^2 is the set of base points of a pencil of E^3's, its associated Q_9^5 is the set of base points of a pencil of E^6's on a V_2^4, the map of the plane by conics.*

This is an immediate consequence of the elementary theorem that if three of the points of such a planar set are on a line the remaining six are on a conic.

We observe that for such a pencil of E^6's on a given V_2^4 each E^6 according to Theorem 18 is contained on three other V_2^4's whence this special Q_9^5 is on ∞^1 V_2^4's one of which is isolated while the others divide into triads which depend rationally on a parameter.

If 8 points of such a special Q_9^5 are given, the locus of the 9th is a 3-way, four of whose points are on any E^6 through the given 8 points. This 3-way is the extension of the Weddle surface and bears the same relation to the hyperelliptic functions of genus three as the Weddle to those of genus two. This relation will be discussed in a forthcoming paper.

If P_9^2 of Theorem 20 is the set of flex points of an E^3, the base points of a syzygetic pencil, then any two are on a line with a third, whence

THEOREM 21. *There exists in S_5 a set of 9 points invariant under a Hesse collineation G_{216} with the property that any two points determine a third such that the remaining six are on an S_4. The configuration contains 12 S_4's, eight on each point.*

This set of 9 points has the unusual property that if six be selected which form a reference 6-point, no other one can be taken to be the unit point, since each of the other three must lie in one of the reference S_4's. Using a proper set of six as reference points the coördinates of the other three are

$$\begin{array}{cccccc} \omega, & \omega^2, & -1, & -\omega^2, & 0, & -\omega; \\ -1, & 1, & \omega^2, & -1, & -\omega^2, & 0; \\ 1, & -1, & \omega, & 0, & -\omega, & -1 \end{array} \qquad (\omega = e^{2\pi i/3}).$$

The problem of obtaining the four surfaces V_2^4 on a given Q_9^5 may be solved through the use of the associated set P_9^2 as follows:

THEOREM 22. *On the E^3 on P_9^2 join the 9th base point of the pencil on $p_1, \cdots, p_8$ to p_9 to meet E^3 again in p'. From p' draw a tangent to E^3 at p'' (4 choices). Construct a set $r_9, r_1, \cdots, r_8$ congruent to $p'', p_1, \cdots, p_8$ under J^5 with centers r_9, p''. Then conics map the set R_9^2 upon the set Q_9^5 associated to P_9^2 and map the plane upon one of the four V_2^4's on Q_9^5.*

We now consider sets of 9 and of 10 points in S_5 with reference to the normal surfaces M_2^4 and the rational sextic curves R^6. We have noted that on Q_9^5 there are ∞^2 M_2^4's and ∞^2 R^6's. Only ∞^1 of the M_2^4's contain the unique E^6 on Q_9^5. For the M_2^4 mapped from the plane by cubic curves on the base O^2, σ contains ∞^8 E^6's which are mapped from quartic curves with nodes at O, σ whence M_2^4 and E^6 on it have 37 constants. But E^6 alone has 36 constants, whence on E^6 there are ∞^1 M_2^4's. These are the bisecants of the ∞^1 involutions $u + u' \equiv k$, since lines on O cut out such an involution on a ternary quartic with node at O. An M_2^4 on Q_9^5 and not containing E^6 can have no other point in common with E^6. For if E^6 were to meet M_2^4 in 10 points, at least four of

the quadrics on M_2^4 would contain E^6. But four such quadrics meet in a residual conic. We now prove

THEOREM 23. *The locus of the* ∞^2 M_2^4*'s on* Q_9^5 *is a cubic spread with the* E^6 *on* Q_9^5 *for double curve. A point of this cubic spread forms with* Q_9^5 *a symmetrical set* Q_{10}^5 *which are the meets of two* M_2^4*'s and whose associated set* P_{10}^3 *is on a quadric surface. The cubic spread is that locus of* ∞^2 *trisecant planes of* E^6 *whose meets with* E^6 *lie with* Q_9^5 *on a quadric.*

For the condition that P_{10}^3 is on a quadric surface is of degree two in each point p_i and therefore is a sum of products of 5 four-row determinants. The corresponding condition on Q_{10}^5 is a sum of products of 5 six-row determinants and therefore is of degree three in each point q_i. Since the condition on P_{10}^3 is invariant under regular Cremona transformation this is likewise true of Q_{10}^5. Hence if q_{10} is variable the cubic spread must have nodes at Q_9^5. According to Theorem 13, Q_{10}^5 is the set of points of intersection of two M_2^4's on Q_9^5. Since the cubic spread contains the ∞^1 M_2^4's on Q_9^5 which contain E^6, it contains the bisecant locus B_3^9 of E^6 and therefore is a member of the pencil of Theorem 18 and contains E^6 as a double curve. To prove the trisecant plane property we observe (and omit the verification) that if a quadric contains M_2^4, a plane on this quadric meets M_2^4 in a point. Given then an M_2^4 and a plane trisecant to E^6 at v_1, v_2, v_3 such that $u_1 + \cdots + u_9 + v_1 + v_2 + v_3 \equiv 0$, of the 6 quadrics on M_2^4 and therefore on Q_9^5 at least four are on v_1, v_2, v_3 and at least one contains the plane $v_1 v_2 v_3$ which therefore meets M_2^4 in a point. As M_2^4 varies in the ∞^2 system on Q_9^5, this point runs over the trisecant plane.

An R^6 on Q_9^5 is on a unique M_2^4 on Q_9^5 and vice versa. For given the M_2^4 mapped by cubics on O^2, σ the R^6's are mapped from ternary quintics with 4-fold point at O and simple point at σ, whence on Q_9^5 there is a unique R^6. These R^6's meet the generators in one point and the directrix conics in four points whose four parameters on conic and on R^6 are projective. Given R^6 on Q_9^5, its quadrisecant planes each carry a unique conic with the projective 4-point property just mentioned and the locus of these conics is the unique M_2^4 on Q_9^5 and R^6.

If in the proof of Theorem 12 the point σ' is on a ternary quintic which maps into an R^6 on Q_9^5, then σ coincides with O in some direction and the set P_{10}^3 is on a nodal quadric. For such a set the two M_2^4's coincide. Hence

THEOREM 24. *The two conditions that* Q_{10}^5 *be on an* R^6 *are that its associated* P_{10}^3 *be on a nodal quadric. On such a* Q_{10}^5 *there is but one* M_2^4.

The ∞^1 M_2^4's on Q_9^5 which contain the E^6 on Q_9^5 are obtained by mapping P_9^2 on Q_9^5 in the ∞^1 ways described in § 1. All of the ∞^2 M_2^4's on Q_9^5 are obtained by the following construction.

THEOREM 25. *For the set* P_9^2 *we choose a center* p *(in* ∞^2 *ways) and, for arbitrarily chosen* p_{10}*, construct a set* $r_1, \cdots, r_9, \sigma, O$ *congruent to* $p_1, \cdots,$

p_9, p_{10}, p under J^6 with centers O, p. Then cubics on $O^2\sigma$ map $r_1, \cdots, r_9$ upon the set Q_9^5 associated to P_9^2, and map the plane upon one of the ∞^2 M_2^4's on Q_9^5.

For if p_1, p_2, p_3 are on a line, then $r_4, \cdots, r_9, \sigma, O^2$ are on a cubic or $q_4, \cdots, q_9$ are on an S_4. That these M_2^4's are all distinct follows from the fact that the ∞^2 line pencils from p to P_9^2 are projectively distinct. We observe that, when p has been chosen and thereby an M_2^4 isolated, the variation of p_{10} implies the variation of the point $\overline{\sigma O}$ of M_2^4 over the M_2^4.

We shall close with an application to the sets of 10 nodes of a rational plane sextic and of a symmetroid quartic surface Σ. These two figures are related as follows. The sextic $S(t)$ has a conjugate rational sextic $R(t)$ in space such that the plane sections of the one are apolar to the line sections of the other. The locus of planes which cut $R(t)$ in catalectic sextics is Σ (as an envelope) and the 10 planes which cut $R(t)$ in cyclic sextics (reducible to a sum of two sixth powers) are the ten double planes of Σ. If such a cyclic sextic is $(p_1 t)^6 + (p_2 t)^6 = 0$, then $(p_1 t)\cdot(p_2 t) = 0$ are the nodal parameters of a double point of $S(t)$. Thus the nodes of Σ and the nodes of $S(t)$ are in correspondence. It is known that there are two projectively distinct rational sextics $S(t)$, $S(\tau)$ which determine the same Σ. I have proved but not yet published the fact that if 6 nodes of $S(t)$ are on a conic then the complementary 4 nodes of Σ are on a plane. Hence conics on the plane map the plane on a V_2^4 in S_5 and the ten nodes of $S(t)$ upon a Q_{10}^5 on V_2^4 which is associated to the P_{10}^3 of nodes of Σ. But also conics of the plane of $S(\tau)$ map this plane on a $V_2'^4$ and the ten nodes of $S(\tau)$ upon the same Q_{10}^5, since this set also is associated to P_{10}^2. From this there follows at once

THEOREM 26. *If two Veronese surfaces V_2^4, $V_2'^4$ meet in 10 points, Q_{10}^5, then this set is associated to the set P_{10}^3 of nodes of a Cayley symmetroid. The spreads V_4^3, $V_4'^3$, with double V_2^4, $V_2'^4$ respectively, each cut the double spread of the other in a 12-ic curve with nodes at Q_{10}^5. These curves are the maps from the plane of the two rational plane sextics associated with the symmetroid.*

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ON ALGEBRAIC FUNCTIONS WHICH CAN BE EXPRESSED IN TERMS OF RADICALS*

BY

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I. Introduction

We consider, in this paper, an irreducible algebraic relation

$$F(w, z) = 0,$$

of degree n in w and of genus p, and seek to determine those cases in which w can be expressed in terms of z by means of radicals. Why the results obtained here should not have been found before this time is a question which has puzzled us as much as it will puzzle the reader.

Our results for the case in which n is prime are fairly complete. We show first that *if n is prime, and if w can be expressed in terms of radicals, then for every genus except zero there exists an upper bound for n.* For any given genus the mechanisms of the possible Riemann surfaces for w can be determined in a finite number of steps.

For the case of genus zero our problem becomes that of finding all rational functions whose inverses can be expressed in terms of radicals. Given one such function, others can be obtained by performing linear transformations upon the function and upon the variable. We show that all rational functions of prime degree with inverses expressible in terms of radicals can be thus obtained from functions of the following types:

(*a*) *The powers of w.*

(*b*) *The polynomials which occur in the formulas for the multiplication of the argument in the function* $\cos u$.

(*c*) *The fractional rational functions which occur in the formulas for the transformations of the periods of the function* $\wp u$.

(*d*) *For the case of* $n \equiv 1 \pmod 4$, *the fractional rational functions met, in the lemniscatic case* $(g_3 = 0)$, *in the formulas for the multiplication of the argument of* $\wp^2 u$ *by* $\alpha \pm \beta i$, *where* $\alpha^2 + \beta^2 = n^2$.

(*e*) *For the case of* $n \equiv 1 \pmod 6$, *the fractional rational functions met, in the equianharmonic case* $(g_2 = 0)$, *in the formulas for the multiplication of the arguments of* $\wp' u$ *and* $\wp'^2 u$ *by* $\alpha \pm \beta i \sqrt{3}$, *where* $\alpha^2 + 3\beta^2 = n^2$.

The functions just described are already known to have inverses expressible

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in terms of radicals. We prove here that they are the only such rational functions of prime degree.

What we have done for the case in which n is composite is to determine all polynomials whose inverses can be expressed in terms of radicals. In stating our result we employ the terms of our paper *Prime and composite polynomials*.* A polynomial $F(z)$ is there called *composite* if there exist two polynomials, $\phi_1(z)$ and $\phi_2(z)$, each of degree greater than unity, such that $F(z) = \phi_1[\phi_2(z)]$. Otherwise, $F(z)$, if of degree greater than unity, is called prime. Let

$$F = \phi_1 \phi_2 \cdots \phi_r,$$

where each $\phi_i(z)$ is a prime polynomial, and is understood to be substituted for z in the polynomial which precedes it.

We show that *if the inverse of $F(z)$ can be expressed in terms of radicals, each $\phi_i(z)$, if not of degree 4, can be obtained by means of linear transformations either from a prime power of z or from a trigonometric polynomial of prime degree.* We will have, that is, if $\phi_i(z)$ is of degree $m \neq 4$,

$$\phi_i = \lambda_1 \pi \lambda_2,$$

where $\lambda_1(z)$ and $\lambda_2(z)$ are linear and where either $\pi(z) = z^m$ or else $\cos mu = \pi(\cos u)$.

The work for this case, with n composite, consists mainly in the proof of a theorem on substitution groups.

II. Functions with a prime number of values

If the degree n in w of the irreducible algebraic relation

$$F(w, z) = 0 \tag{1}$$

is prime, and if w can be expressed in terms of radicals, the group of monodromy of (1) is either the metacyclic group or one of its transitive subgroups. The group of monodromy must contain a substitution of order n. We number the branches of w in such a way that the group contains the substitution $(1\ 2 \cdots n)$, and represent the metacyclic group with the formula

$$\nu' \equiv a\nu + b \pmod{n} \quad \begin{pmatrix} a = & 1, 2, \cdots, n-1 \\ b = 0, & 1, 2, \cdots, n-1 \end{pmatrix}. \tag{2}$$

The non-identical substitutions with $a = 1$ displace every index ν. The substitutions with $a \neq 1$ leave a single index fixed. If a substitution of the metacyclic group consists of more than one cycle, its cycles are all of the same order.

Suppose that w has q critical points. We consider the elementary substitutions of the group of monodromy, which correspond to single turns

* These Transactions, vol. 23 (1922), p. 51.

around these critical points. Suppose that there are α of them with $a = 1$ and $q - \alpha$ with $a \neq 1$. We designate the orders of the latter by $s_1, s_2, \cdots, s_{q-\alpha}$.

If the genus of (1) is p, we have, according to the well known formula of Riemann,

$$(n-1)\alpha + \sum_{i=1}^{i=q-\alpha} \frac{n-1}{s_i}(s_i - 1) = 2(n-1) + 2p,$$

or

$$\sum_{i=1}^{i=q-\alpha} \frac{1}{s_i} = q - 2 - \frac{2p}{n-1}. \tag{3}$$

Since no s_i is less than 2, the first member of (3) is not greater than $q/2$, and we find from (3)

$$q \leqq 4 + \frac{4p}{n-1}, \tag{4}$$

which shows that when p is given, there exists an upper bound for q, independent of n.

We shall prove now that *if p is not zero, there exists an upper bound for n which depends only on p.*

Suppose first that $q > 4$. We have then from (4),

$$n - 1 \leqq \frac{4p}{q-4} \leqq 4p.$$

If $q = 4$, equation (3) gives

$$\frac{2p}{n-1} = 2 - \sum \frac{1}{s_i}.$$

It is seen quickly that the second member of this last equation is at least equal to $1/6$. We have thus, in this case,

$$n - 1 \leqq 12p.$$

If $q = 3$, equation (3) gives

$$\frac{2p}{n-1} = 1 - \sum \frac{1}{s_i}.$$

The three integers the sum of whose reciprocals is less than unity by as small a positive number as possible are 2, 3 and 7. We have thus for this case

$$n - 1 \leqq 84p.^*$$

A closer examination of the problem would lead to smaller bounds for n than those found above.

On being given p, we can, with the help of equation (3) and of the upper

* If $q - \alpha < 3$, we have a stronger inequality.

bounds for n and q, determine the possible Riemann surfaces for w in a finite number of steps. We shall not follow this question further in the present paper.

We consider the case of $p = 0$, which includes the most interesting examples already known of algebraic functions expressible in terms of radicals. We must have $q = 2, 3$ or 4.

Following our paper referred to in the introduction, we shall call the sum of the orders of the branch points of an algebraic function at a given point the *index* of the function at that point. The sum of the indices of the inverse of a rational function of degree n is $2n - 2$.

If, when we represent the substitution at a critical point in the form (2), the coefficient a belongs to the exponent d modulo n, where $d > 1$, the substitution is of order d, and the index at the critical point is $(n - 1)(d - 1)/d$.

We shall consider first those cases in which the Riemann surface for w has a branch point of order $n - 1$. Such a branch point must be present at infinity if w is the inverse of a polynomial. As the index of w at any critical point is at least $(n - 1)/2$, and as the sum of the indices of w is $2n - 2$, we must have, in this case, $q = 2$ or $q = 3$.

If $q = 2$, w must have two branch points of order $n - 1$. Subjecting z to a suitable linear transformation, we may suppose that one of these points is at infinity and the other at zero. The surface thus obtained is recognized as that for $w = z^{1/n}$. The functions uniform on it are rational functions of $z^{1/n}$. Of these, the only ones which are inverses of polynomials are linear integral functions of $z^{1/n}$.

If $q = 3$, the remaining two critical points of w, since each has an index not less than $(n - 1)/2$, must each have precisely $(n - 1)/2$ as index. Subjecting z to a suitable linear transformation, we may suppose that the branch point of order $n - 1$ is at infinity, and the other two critical points at $z = 1$ and $z = -1$ respectively. The substitutions at the latter points are of the form

$$\nu' \equiv -\nu + h_1, \qquad \nu' \equiv -\nu + h_2 \qquad (\text{mod } n),$$

respectively. The substitution at infinity, the result of following the second of these two by the first, is

$$\nu' \equiv \nu + h_1 - h_2 \qquad (\text{mod } n).$$

As this substitution is of period n, we have $h_1 \not\equiv h_2$. If we renumber the branches of w, giving to the branch numbered ν the number μ determined by the congruence

$$\nu \equiv (h_1 - h_2)\mu + \frac{n+1}{2} h_1 \qquad (\text{mod } n), \tag{5}$$

the three elementary substitutions become

$$\mu' \equiv -\mu, \qquad \mu' \equiv -\mu - 1, \qquad \mu' \equiv \mu + 1 \qquad (\text{mod } n).$$

Thus only one mechanism is possible for the surface of w. Now it is well known that the trigonometric polynomial of degree n, $f_n(w)$, defined by the relation

$$\cos nu = f_n(\cos u)$$

has an inverse expressible in terms of radicals, the critical points of the inverse being at 1, -1, and ∞. Hence w must be a rational function of $f_n^{-1}(z)$.

Summarizing the foregoing results, we see that *the only polynomials of prime degree whose inverses can be expressed in terms of radicals are those of the forms* $a(w+b)^n + c$ *and* $af_n(bw+c)+d$, *where* $\cos nu = f_n(\cos u)$.

We consider now the case of $q = 4$. As the index of w at each critical point is at least $(n-1)/2$ and as the sum of the indices is $2n-2$, the index of w is precisely $(n-1)/2$ at each critical point. The corresponding substitutions are all of order 2. Subjecting z to a suitable linear transformation, we may throw one of the critical points to ∞, and so dispose the others that the sum of their affixes e_1, e_2, and e_3 is zero. Let the substitutions at e_1, e_2, and e_3 be respectively

$$\nu' \equiv -\nu + h_1, \qquad \nu' \equiv -\nu + h_2, \qquad \nu' \equiv -\nu + h_3 \pmod{n}.$$

Suppose that h_1 and h_2 are unequal. If we give to the branch of w numbered ν the number μ determined by (5), the three substitutions become

$$\mu' \equiv -\mu, \qquad \mu' \equiv -\mu - 1,$$
$$(h_1 - h_2)\mu' \equiv (h_2 - h_1)\mu + h_3 - h_1 \pmod{n}.$$

As h_3 varies from 0 to $n-1$, we obtain n types of surfaces which, it will be seen below, are all distinct.

If h_1 and h_2 are equal, they must be distinct from h_3, else the surface would not hang together. We can in this case reduce the three substitutions to

$$\mu' \equiv -\mu, \qquad \mu' \equiv -\mu, \qquad \mu' \equiv -\mu - 1 \pmod{n}.$$

Thus, the critical points being disposed as described above, there are at most $n+1$ distinct surfaces for w. To identify these, we construct the elliptic function $\wp(u|\omega_1, \omega_3)$, with $\wp(\omega_i) = e_i$ $(i = 1, 2, 3)$. This is possible, since $e_1 + e_2 + e_3 = 0$. Let

$$(6) \qquad \begin{aligned} \Omega_1 &= a\omega_1 + b\omega_3, \\ \Omega_3 &= c\omega_1 + d\omega_3, \end{aligned}$$

where $ad - bc = n$. It is well known that there are $n+1$ distinct transformations (6), and that for every transformation, we have

$$\wp(u|\omega_1, \omega_3) = R[\wp(u|\Omega_1, \Omega_3)],$$

where $R(w)$ is a fractional rational function of degree n, whose inverse can

be expressed in terms of radicals. The critical points of $R^{-1}(z)$ will correspond to those values which $\wp(u|\omega_1, \omega_3)$ assumes twice at a point, namely, e_1, e_2, e_3, and ∞. It is easy to show that the $n+1$ Riemann surfaces for the inverses of the functions $R(w)$, occurring in the $n+1$ distinct transformations (6), have distinct mechanisms. The $n+1$ surfaces exhibited above can be none other than these.

We pass finally to the case in which $q = 3$ and in which no branch point of order $n-1$ exists. We must have, by (3),

$$\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} = 1.$$

There are the three possibilities:

(a) $s_1 = s_2 = \frac{1}{4}, \quad s_3 = \frac{1}{2};$

(b) $s_1 = s_2 = s_3 = \frac{1}{3};$

(c) $s_1 = \frac{1}{2}, \quad s_2 = \frac{1}{3}, \quad s_3 = \frac{1}{6}.$

Consider Case (a). As every s_i is a divisor of $n-1$, we must have $n \equiv 1 \pmod 4$. Subjecting z to a suitable transformation, we can place the critical points with substitutions of order 4 at 0 and ∞ and the third critical point at any point e_1^2. Normalizing the substitutions at the critical points, we find that there are not more than two distinct mechanisms for the surface. We now take $\wp u$ so that $\wp(\omega_1) = e_1$, $\wp(\omega_2) = 0$, and $\wp(\omega_3) = -e_1$. This corresponds to the lemniscatic case. Now, as $n \equiv 1 \pmod 4$, we have $n = \alpha^2 + \beta^2$, where α and β are integers. In the lemniscatic case, the two functions $\wp^2(\alpha \pm \beta i)u$ are rational functions of $\wp^2 u$.* The rational functions thus obtained can be inverted in terms of radicals. It is not hard to identify their surfaces with those of Case (a).

Similarly, it is found that Cases (b) and (c) lead to the rational functions mentioned in Case (e) of the introduction, which are met in the multiplication formulas for the equianharmonic case.†

III. Polynomials of composite degree

We consider a polynomial $w = F(z)$, of prime or composite degree n, and seek those cases in which $F^{-1}(w)$, the inverse of $F(z)$, can be expressed in terms of radicals.‡

* For details relative to complex multiplication in the lemniscatic and equianharmonic cases, see Ritt, *Periodic functions with a multiplication theorem*, these Transactions, vol. 23 (1922), p. 16.

† A detailed discussion of all Riemann surfaces considered in this section is contained in a paper by the writer, *Permutable rational functions*, now in the hands of the editors of these Transactions.

‡ We have interchanged the rôles of w and z in order to conform with the notation of our paper referred to above.

If $F(z)$ is a composite polynomial, that is, if

$$F(z) = \phi_1[\phi_2(z)],$$

where $\phi_1(z)$ and $\phi_2(z)$ are of degrees greater than unity, then if $F^{-1}(w)$ can be expressed in terms of radicals, $\phi_1^{-1}(w)$ and $\phi_2^{-1}(w)$ can also be so expressed, for

$$\phi_1^{-1}(w) = \phi_2[F^{-1}(w)], \qquad \phi_2^{-1}(w) = F^{-1}[\phi_1(w)].$$

We may thus restrict ourselves to the determination of prime polynomials whose inverses can be expressed in terms of radicals.

We will show that if the inverse of a *prime* polynomial is expressible in terms of radicals, the degree of the polynomial, if not equal to four, is a prime number. Thus, using the result found for polynomials in the preceding section, we will know that *if $F(z)$ has the decomposition into prime polynomials*

$$F = \phi_1\,\phi_2 \cdots \phi_r$$

*each $\phi_i(z)$ is either of degree 4, or else is of the form $\lambda_1\,\pi\lambda_2$, where $\lambda_1(z)$ and $\lambda_2(z)$ are linear and where $\pi(z)$ is either a prime power of z or a trigonometric polynomial of prime degree.**

We refer to § II of our paper, *Prime and composite polynomials*, for a proof of the fact that *a necessary and sufficient condition that a polynomial be prime is that the group of monodromy of its inverse be primitive.*

It is well known that the degree of a primitive solvable group is a power of a prime. As the substitution corresponding to the branch point at infinity of the inverse of a polynomial of degree n consists of a single cycle of n letters, the proof that prime polynomials whose inverses can be expressed in terms of radicals are either of prime degree or of degree 4 will be complete as soon as we have proved the following theorem on substitution groups:

THEOREM. *A primitive solvable group in p^m letters with p prime and $m > 1$ cannot contain a substitution of order p^m, except in the case of $p = 2$, $m = 2$.*

Let G be a primitive solvable group of degree p^m. Suppose that G contains a cyclic subgroup C of order p^m.

It is well known that G contains an invariant transitive abelian subgroup Γ, of order p^m, every substitution of which, except identity, is of order p. As Γ is permutable with C, these two groups generate a group H in G, the order of which is the product of the orders of Γ and C, p^{2m}, divided by the order of the group of substitutions common to Γ and C. The order of H is a power of p greater than p^m. The substitutions of H are all of the form $c\gamma$ where c and γ are substitutions of C and Γ respectively.

The group C is invariant in a subgroup of H of order greater than p^m.

*Certainly if each $\varphi_i(z)$ is of one of the three types described, $F^{-1}(w)$ can be expressed in terms of radicals.

Hence there must be substitutions of H which are not in C, and with respect to which C is invariant. Suppose that $c\gamma$ is such a substitution, where γ is not in C. Then, since

$$\gamma^{-1} c^{-1} C c\gamma = \gamma^{-1} C\gamma = C,$$

C is invariant with respect to certain substitutions γ of Γ, which are not in C.

Let

$$c_1 = (0\ 1\ 2 \cdots \nu \cdots p^m - 1)$$

be the substitution which generates C. We shall determine the group of substitutions which converts c_1 into a power of itself. Let α be a substitution such that

$$\alpha^{-1} c_1 \alpha = c_1^r,$$

where r is any integer not divisible by p. It is evident that α is determined as soon as r, and the index s by which α replaces 0, are given. Consider the substitution given analytically by

$$\nu' \equiv r\nu + s \pmod{p^m}. \tag{7}$$

Noting that c_1 has the representation $\nu' \equiv \nu + 1 \pmod{p^m}$, we see that $\alpha^{-1} c_1 \alpha$ has the representation $\nu' \equiv \nu + r \pmod{p^m}$. Thus α transforms c_1 into c_1^r and replaces 0 by s. The group in which C is invariant is given by (7), where r assumes all values prime to p, and where s is unrestricted.

We shall now impose the condition that a substitution of the form (7) belong to Γ, but not to C. We have for α^p the representation

$$\nu' \equiv r^p \nu + s(r^{p-1} + r^{p-2} + \cdots + 1) \pmod{p^m}.$$

As α is not in C, we cannot have $r \equiv 1 \pmod{p^m}$. Since α belongs to Γ, α^p is identity, so that

$$r^p \equiv 1 \pmod{p^m}, \tag{8}$$

$$s(r^{p-1} + r^{p-2} + \cdots + 1) \equiv 0 \pmod{p^m}. \tag{9}$$

By Fermat's theorem,

$$r^p \equiv r \pmod{p}, \tag{10}$$

so that, by (8) and (10), $r \equiv 1 \pmod{p}$. Let $r = kp + 1$. We have

$$r^{p-i} \equiv (p - i)kp + 1 \pmod{p^2},$$

so that

$$r^{p-1} + r^{p-2} + \cdots + 1 \equiv \sum_{i=1}^{i=p} [(p - i)kp + 1] \pmod{p^2},$$

$$\equiv \frac{kp^2(p - 1)}{2} + p \pmod{p^2}.$$

Suppose that $p > 2$. Then $p - 1$ is even and

(11) $$r^{p-1} + r^{p-2} + \cdots + 1 \equiv p \pmod{p^2}.$$

That is, the first member of (11) is divisible by p, but not by p^2. Hence, referring to (9), we see that s is divisible by p^{m-1}.

Suppose then that α has the form

$$\nu' \equiv (kp + 1)\nu + lp^{m-1} \pmod{p^m},$$

where kp is not divisible by p^m. Since Γ is regular, if we can show that α leaves certain indices fixed, we will know that α cannot belong to Γ. Consider the congruence

$$(kp + 1)\nu + lp^{m-1} \equiv \nu \pmod{p^m},$$

or

(12) $$kp\nu + lp^{m-1} \equiv 0 \pmod{p^m}.$$

Since kp is not divisible by p^m, we see that (12) must have roots.

We conclude that for α to belong to Γ and not to C, we must have $p = 2$.

If $p = 2$, we must have, by (8),

$$r^2 \equiv 1 \pmod{2^m}.$$

If $m > 2$, this congruence has the four solutions

$$r \equiv \pm 1, \qquad r \equiv 2^{m-1} \pm 1 \pmod{2^m}.$$

Suppose that $m > 2$, and that $r \equiv 2^{m-1} + 1 \pmod{2^m}$. The highest power of 2 by which $r + 1$ is divisible is the first. Since, by (9), $s(r + 1)$ is divisible by 2^m, we see that s is divisible by 2^{m-1}. That is, α has the form

$$\nu' \equiv (2^{m-1} + 1)\nu + 2^{m-1} l \pmod{2^m}.$$

Since this substitution leaves the index 0, or the index 1, fixed, according as l is even or odd, we cannot have $r \equiv 2^{m-1} + 1 \pmod{2^m}$, if $m > 2$.

Suppose that $m > 2$, and that $r \equiv 2^{m-1} - 1 \pmod{2^m}$. Then, by (9), s must be even. Putting $\nu' = \nu$, we have

$$(2^{m-1} - 2)\nu + 2l \equiv 0 \pmod{2^m}.$$

Now, since $m > 2$, $2^{m-1} - 2$ is divisible by no higher power of 2 than the first, so that the congruence above has roots, and α cannot belong to Γ.

We consider finally the case of $r \equiv -1 \pmod{2^m}$. We have the 2^m substitutions

(13) $$\nu' \equiv -\nu + s \pmod{2^m},$$

which transform C into itself. If s is even, (13) leaves two letters fixed and cannot belong to Γ. Consider those substitutions for which s is odd. We say that if Γ contains one of them, it contains all of them. For if,

in the substitutions

$$\nu' \equiv -\nu + s_1, \qquad \nu' \equiv -\nu + s_2 \qquad (\text{mod } 2^m),$$

s_1 and s_2 are both odd, the substitution $\nu' \equiv \nu + (s_2 - s_1)/2$ (mod 2^m), which belongs to C, transforms the first into the second. As Γ is an invariant subgroup, if either of these substitutions belongs to Γ the other does also.

Suppose that Γ contains the two substitutions

$$\nu' \equiv -\nu + 1, \qquad \nu' \equiv -\nu - 1 \qquad (\text{mod } 2^m),$$

which we denote by α_1 and α_2 respectively. As Γ is abelian, we have, equating $\alpha_1 \alpha_2$ and $\alpha_2 \alpha_1$,

$$\nu - 2 \equiv \nu + 2 \qquad (\text{mod } 2^m),$$

from which it follows that $m = 2$.

In the case of $p = 2$, $m = 2$, the symmetric group in four letters has substitutions of order 4.

The proof of the theorem is completed.

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ON THE LOCATION OF THE ROOTS OF THE JACOBIAN OF TWO BINARY FORMS, AND OF THE DERIVATIVE OF A RATIONAL FUNCTION*

BY

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1. **Introduction.** There have recently been published the following results:†

THEOREM I. *If the points* z_1, z_2, z_3 *vary independently and have circular regions as their respective loci, then the locus of the point* z_4 *defined by the real constant cross ratio*

$$\lambda = (z_1, z_2, z_3, z_4)$$

is also a circular region.

THEOREM II. *Let* f_1 *and* f_2 *be binary forms of degrees* p_1 *and* p_2 *respectively, and let the circular regions* C_1, C_2, C_3 *be the respective loci of* m *roots of* f_1, *the remaining* $p_1 - m$ *roots of* f_1, *and all the roots of* f_2. *Denote by* C_4 *the circular region which is the locus of points* z_4 *such that*

$$(z_1, z_2, z_3, z_4) = \frac{p_1}{m},$$

when z_1, z_2, z_3 *have the respective loci* C_1, C_2, C_3. *Then the locus of the roots of the jacobian of* f_1 *and* f_2 *is composed of the region* C_4 *together with the regions* C_1, C_2, C_3, *except that among the latter the corresponding region is to be omitted*‡ *if any of the numbers* $m, p_1 - m, p_2$ *is unity. If a region* C_i $(i = 1, 2, 3, 4)$ *has no point in common with any other of those regions which is a part of the locus of the roots of the jacobian, it contains precisely* $m - 1$, $p_1 - m - 1$, $p_2 - 1$, *or* 1 *of those roots according as* $i = 1, 2, 3$, *or* 4.

It is the primary object of the present paper to consider extensions of Theorem II in various directions. Chapter I studies the possibility of extend-

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† These Transactions, vol. 22 (1921), pp. 101–116; this paper will be referred to as II. It was preceded by a paper, these Transactions, vol. 19 (1918), pp. 291–298, which will be referred to as I, and was followed by a third paper, these Transactions, vol. 23 (1922), pp. 67–88, which will be referred to as III. We shall also have occasion to refer to two other of our papers, using the letters A and S respectively: Annals of Mathematics (2), vol. 22 (1920), pp. 128–144; *Comptes Rendus du Congrès International des Mathématiciens*, Strasbourg, 1920, pp. 349–352. The term *locus* in Theorem I of the present paper replaces the term *envelope* used in II.

‡ The corresponding region is to be omitted in this *enumeration* of the points of the locus; the corresponding region may nevertheless be in whole or in part, a portion of the locus of the roots of the jacobian.

ing Theorem II to include regions which are not circular. It is found (Theorem III) that the reasoning formerly used cannot be directly extended, and specific examples bring out the nature of the difficulties in supplying other modes of reasoning. Chapter II treats the extension of Theorem II by increasing the number of circular regions which are allowed to be loci of roots of the ground forms of the jacobian. Theorems VI and XI are fairly general results obtained by this extension, the principal results of the paper. Chapter III is a short chapter which deals with centers of gravity; the results are mainly generalizations of well known results for polynomials and their derivatives. Finally, Chapter IV deals with the case of the roots of the jacobian of real forms. Theorem XV is a rather general result which applies to the roots of the derivative of a polynomial which has only real roots.

Chapter I: On the extension of Theorem II to other than circular regions

2. **A distinctive property of circular regions.** We shall now undertake to consider the extension of Theorem II to regions which are not circular. Inasmuch as rather large extensions of Theorem I in this direction can be obtained without difficulty, and in fact have been obtained in III, our first tendency is to attempt to extend Theorem II by repeating our previous reasoning of II, p. 113. This turns out to be impossible, due to the failure of Lemma I (II, p. 102) to admit of large extension to other than circular regions:

Theorem III. *If a closed region C has the property that the force at any external point P due to every set of k unit particles in C is equivalent to the force at P due to k unit particles coinciding at some point of C, then C is a circular region.*

In proving this theorem we do not need to assume the property stated for every k, but merely for any one particular k (except of course $k = 1$, for which the result is absurd). We shall suppose $k = 2$ and leave to the reader the modifications for the other values of k.

If C is the whole plane there is no external point P and we may consider the theorem true, since C is a circular region. Similarly, if C is a single point the theorem is true. In the sequel we assume C to be neither the entire plane nor a single point.

It will be noted that the force at P due to equal particles at two points M and N is equivalent to the force at P due to two coincident particles situated at Q, the harmonic conjugate of P with respect to M and N,* which situation is invariant under linear transformation. We shall proceed to prove the

Lemma. *If P is exterior to C, and M and N are two points of C, then C contains every point on that arc of the circle MNP bounded by M and N which does not contain P.*

* This fact is quite easy to prove; see for example A, pp. 128–129.

Choose P at infinity and equal particles at M and N; the point Q which is the mid point of the line segment MN is seen to be in C. Then the mid points of the segments MQ, QN are also in C, and in fact we have a set of points in C everywhere dense on the segment MN. Hence this entire segment belongs to C.

The region C contains at least two points. It follows from the lemma that C contains an infinity of points. Transform one point R of the boundary of C to infinity, and consider two other distinct points V and W of that boundary. We proceed to prove that every point of the segment VW is a point of C. We can find a sequence of points not belonging to C but approaching R. Hence there is a sequence of points (the harmonic conjugates of the former sequence with respect to V and W) belonging to C and approaching the mid point U of VW, so U belongs to C. Then the mid points of UV and VW also belong to C, and in this way we prove that every point of VW belongs to C. But R also belongs to C, and hence we can prove that either every point of that infinite segment of RV which does not contain W belongs to C or every point of that infinite segment RW which does not contain V belongs to C; for definiteness suppose the latter.

There exists an infinite sequence of points not belonging to C but approaching W, and we assume as of course we may do that we have oriented the line VW horizontally and that we have an infinite sequence of such points $\{Z_k\}$ all lying in the lower half plane. If a line through Z_k cuts the segment VWR, then every point of that line which lies in the upper half plane belongs to C, by the lemma. For any preassigned point Y in the upper half plane we can choose a point Z_k such that YZ_k cuts the segment VWR, so every point of the upper half plane belongs to C as does also every point of the line VW.

If the region C does not consist of precisely the upper half plane including its boundary, there is a point X of C in the lower half plane. We can make use of the fact that R is a point of the boundary of C as before, and prove that the entire finite segment joining X to an arbitrary point of the line VW belongs to C. Then V and W are not boundary points of C, contrary to our assumption. The demonstration of the theorem is now complete.

Theorem III is quite easily proved, although by essentially the same methods, if we assume C to be bounded by a regular curve. Thus the property considered is invariant under linear transformation, for the position of the k coincident particles is uniquely determined by P and the original particles, and by Theorem I (I, p. 291 = II, p. 101) P is a root of the jacobian of the two binary forms each of degree k and whose roots are respectively the k original particles in C and the k coincident particles. If C is not bounded by a circle, its boundary can be transformed into a contour which is not convex.* Then

* See Theorem III of a note by the present writer, Annals of Mathematics (2), vol. 22 (1921), pp. 262–266.

there are two points A and B on the boundary of C such that the segment AB is exterior to C and such that there is a point P exterior to C and on the line AB but not on the segment AB. The force at P due to one particle at A and $k-1$ particles at B is equal to the force at P due to k coincident particles at a point which is on the segment AB but which coincides with neither A nor B and hence which is exterior to C.

The essence of Lemma I (II, p. 102) was proved and applied by Laguerre;* his formulation of the result was quite different from the present formulation, although the application was to the location of the roots of algebraic equations. Thus Theorem III appears as a sort of converse of the theorem of Laguerre, as well as of Lemma I (II, p. 102).

3. **Successive application of Theorem II to the determination of loci.** The property of circular regions stated in Theorem III seems to be conclusive in showing that the reasoning of II, p. 113, cannot be reproduced to give large extensions of Theorem II. Theorem III justifies, moreover, the somewhat artificial use of circular regions in III, Theorem XIII.

We now point out by two simple examples how results can be found from successive applications of Theorem II. These examples are not given as large extensions of Theorem II, but rather to show how difficult is that extension to regions which are not circular. The proofs of these theorems are left to the reader; the proof of the latter depends on III, Theorem IX.

THEOREM IV. *Suppose we have a finite or infinite number of sets of regions* $C_1^{(n)}, C_2^{(n)}, C_3^{(n)}, C_4^{(n)}$, *of Theorem* I *corresponding to the value* $\lambda = p_1/m$, *and suppose that no* $C_i^{(n)}$ *has a point in common with* $C_j^{(k)}$ *unless* $i = j$ $(i, j = 1, 2, 3, 4)$. *Denote by* T_1, T_2, T_3, T_4, *the regions common to all the* $C_1^{(n)}, C_2^{(n)}, C_3^{(n)}, C_4^{(n)}$, *respectively. Then if* T_1 *contains* m *roots of a bilinear form* f_1, *if* T_2 *contains all the remaining* $p_1 - m$ *roots of* f_1, *and if* T_3 *contains all the* p_2 *roots of a second form* f_2, *then the regions* T_1, T_2, T_3, T_4 *contain all the roots of the jacobian of* f_1 *and* f_2. *No two of the regions* T_1, T_2, T_3, T_4 *have a point in common, and they contain respectively* $m-1, p_1-m-1, p_2-1, 1$ *of the roots of the jacobian.*

There is no reason to suppose that the actual locus of the roots of the jacobian is composed of T_1, T_2, T_3, T_4, when T_1, T_2, T_3 are the loci of roots of the ground forms. But if the regions $C_1^{(n)}, C_2^{(n)}, C_3^{(n)}, C_4^{(n)}$ have the disposition suggested in the first part of § 11 (III) or more generally if T_4 is the locus of the point z_4 determined by its cross ratio p_1/m with the points z_1, z_2, z_3 whose loci are T_1, T_2, T_3, these four regions form that locus, except of course that among these latter three the corresponding region is to be omitted if any of the numbers $m, p_1 - m, p_2$ is unity.

* *Œuvres*, pp. 56–63; p. 59.

THEOREM V. *In the situation of* III, *Theorem* VIII, *suppose the point* P *which is a center of external similitude for every pair of the circles* C_1, C_2, C *to be actually external to all those circles. Denote by* T_1, T_2, T *the portions of the interiors of those circles which lie between two half lines through* P *cutting those circles. Then if* T_1 *and* T_2 *are the respective loci of* m_1 *and* m_2 *roots of a polynomial* $f(z)$, *the regions* T_1, T_2, T *are the loci of the roots of its derivative* $f'(z)$, *except that* T_1 *or* T_2 *is to be omitted if* m_1 *or* m_2 *is unity. If* T_1, T_2, T *are mutually external, they contain respectively* $m_1 - 1$, $m_2 - 1$, 1 *of the roots of* $f'(z)$.

A theorem similar to Theorem V will be obtained by cutting the circles C_1, C_2, C by any convex contour, but no result can generally be stated in this case concerning the actual locus of the roots of $f'(z)$.

In discussing the possibility of the extension of Theorem II by reproducing the reasoning of II, p. 113, we reached the impossibility of extending Lemma I (II, p. 102), by which we may replace the force at an arbitrary point P exterior to a region C due to k particles in C by the force at P due to k coincident particles in C. To obtain certain facts, however, concerning the location of the roots of the jacobian, it may not be necessary to replace the k particles in C by k coincident particles in C *for an arbitrary point* P exterior to C but merely *for certain points* P exterior to C. This fact is notably true if the ground forms are real, as we shall show in Chapter IV; it is also true for certain other cases, as we shall now indicate by an extremely simple example.

We consider the polynomial

$$f(z) = z(z - \alpha_1)(z - \alpha_2),$$

where α_1 and α_2 have as their common locus the interior and boundary of the circle C_1 whose center is $z = 6$ and radius unity. Then $f'(z)$ has a root z_1 which has as its locus the interior and boundary of the circle C_1 and a root z_2 which has as its locus the interior and boundary of the circle C whose center is $z = 2$ and radius $1/3$. Under the given conditions, moreover, z_1 and z_2 remain separate and distinct.

Let us now consider the same polynomial but assign to the roots α_1 and α_2 as their common locus the right-hand semicircular region S_1' of C_1. Any point of the right-hand semicircular region S' of C is by III, Theorem IX, a point of the locus of z_2, and we shall prove that no other point is a point of this locus. Suppose a point $\bar{z}$ to be a point of the locus of z_2; $\bar{z}$ is evidently in or on C. The force at $\bar{z}$ due to particles at α_1 and α_2 is equivalent to the force at $\bar{z}$ due to two particles coinciding at some point α. We know that $\bar{z}$ is in or on C, hence exterior to C_1, so α is in or on C_1. Moreover, $\bar{z}$ is not in the half plane bounded by and lying to the right of the line $x = 6$, and hence α is in that half plane. Then α is in S_1', so $\bar{z}$ is in S'.

The locus of z_1 under the conditions stated includes of course S_1', but other points as well. In fact, if we choose $\alpha_1 = 6 + i$, $\alpha_2 = 6 - i$, the two roots of the real polynomial $f'(z)$ cannot be conjugate imaginary, so z_1 is real but to the left of the point $z = 6$ and therefore not a point of S_1'.

We consider anew the same polynomial and assign the left-hand semicircle S_1'' of C_1 as the locus of α_1 and α_2. Any point of S_1'' is evidently a point of the locus of z_1, and no other point belongs to this locus. For any such point would lie in the semicircle S', which is impossible, according to the theorem of Lucas. Under our conditions the locus of z_2 includes the left-hand semicircle S'' of C, but also other points. We choose as before $\alpha_1 = 6 + i$, $\alpha_2 = 6 - i$, and find z_2 to be real and on or within S' as previously noted. But $z_2 \neq 2$, and hence is not a point of S''.

Results which are large extensions of Theorem II to other than circular regions seem difficult to prove, as is shown by Theorems IV and V, even when the regions involved are common to two or more circular regions. But theorems of a certain type are easily established; we give simply one example:

Let the intersecting circles C_1 and C_2 with centers at the points α_1 and α_2 and radii r_1 and r_2 be the respective loci of m_1 and m_2 roots of a polynomial $f(z)$. Let the region common to C_1 and C_2 be the locus of $f(z)$, which is supposed to have no roots other than those mentioned. Then the roots of $f'(z)$ lie in C_1, C_2, and the region common to the two circles whose centers are the points

$$\frac{(m_2 + m_3)\alpha_1 + m_1\alpha_2}{m_1 + m_2 + m_3}, \qquad \frac{m_2\alpha_1 + (m_1 + m_3)\alpha_2}{m_1 + m_2 + m_3},$$

and whose radii are respectively

$$\frac{(m_2 + m_3)r_1 + m_1 r_2}{m_1 + m_2 + m_3}, \qquad \frac{m_2 r_1 + (m_1 + m_3) r_2}{m_1 + m_2 + m_3}.$$

The region mentioned in this theorem is not the *locus* of the roots of $f'(z)$, for the intersection of the last two circles mentioned in the theorem can never be a root of $f'(z)$. The proof of these facts is left to the reader.

4. **The number of roots of the jacobian in a circular region.** A question concerning the distribution of the roots of the jacobian which is very closely connected with the question of extending Theorem II to regions other than circular is that of the number of roots of the jacobian in the regions C_i when two or more of those regions have common points. Thus it might be supposed that C_1, C_2, C_3, C_4 contain *always* $m, p_1 - m, p_2, 1$ of the roots of the jacobian except for the possibility that this number be exceeded when C_i has one or more points in common with another of the regions. This supposition is false, however, as we proceed to show by an example. Thus consider the circles C_1 and C_2 in the form of Theorem II corresponding to III, Theorem VIII

(i.e., S, Theorem I), such that C intersects C_2 but is exterior to C_1. Let the line of centers of the circles intersect C_1 in A_1 (the intersection nearest C and C_2) and intersect C_2 in A_2 and C in A (the intersections farthest from C_1). When m_1 roots of $f(z)$ coincide at A_1 and m_2 roots at A_2, we have $m_1 - 1$ roots of $f'(z)$ at A_1, $m_2 - 1$ roots at A_2, and one root at A. When the m_2 roots at A_2 move slightly so that they do not all coincide but all remain on C_2 and symmetric with respect to the line $A_1 A_2$, the force corresponding at A and in the neighborhood of A becomes equivalent to the force due to m_2 particles coinciding *exterior* to C_2, since these particles can be considered to lie in the circular region consisting of the circle C_2 and all exterior points. Hence the root of $f'(z)$ at A moves and becomes exterior to C; the m_2 roots at A_2 remain in the vicinity of A_2 and there is no root of $f'(z)$ on or interior to C.

The question which we raised has thus been answered so far as concerns a region C which contains no roots of the ground forms. The result is essentially the same for a region which does contain a number of roots of the ground forms. Consider the case of the derivative of a polynomial, the second theorem on page 115 of II, locate m_1 roots of $f(z)$ at the null circle C_1, and locate the remaining m_2 roots at two points A and B different from C_1. There are but two roots z_1 and z_2 of $f'(z)$ distinct from A, B, and C_1, and these are interior to the triangle ABC_1, so a circle C_2 can be drawn which includes A and B (that is, m_2 roots of $f(z)$) but includes neither z_1 nor z_2 and hence contains only $m_2 - 2$ roots of $f'(z)$.

The question we have been considering is closely connected with the following:* Suppose a circle C contains at least r roots of a polynomial $f(z)$ of degree n. What can be said of the number of roots of $f'(z)$ in C?

On the one hand, C may contain all the roots of $f(z)$ and hence all the $n - 1$ roots of $f'(z)$. On the other hand, C may contain r roots of $f(z)$ and yet no root of $f'(z)$ if merely $r < n$. In fact, we prove that C may contain precisely $n - 1$ roots of $f(z)$ and contain a preassigned number p of the roots of $f'(z)$, if merely $p < n - 1$. Locate one root of $f(z)$ at a point P and the other $n - 1$ roots at $n - p - 1$ distinct points which lie on a line L not passing through P. Then we can describe a circle C which includes the $n - p - 1$ distinct points on L and hence p roots of $f'(z)$ but which contains no other roots of $f'(z)$.

* Still another allied question is: Suppose a circle C is known to contain at least r roots of a polynomial of degree n; determine the smallest (concentric) circle C' which always contains at least m roots of the derived polynomial.

The circle C' exists only if $m < r$. For $n = r = m + 1$, the answer is given by Lucas's Theorem. For the case $r = 2$, the circle C' is readily determined by means of a theorem due to Grace, to which reference is made in A, § 4. For the case $n - 1 = r = m + 1$, the circle C' is easily found by the second theorem of II, p. 115. For other cases the problem seems considerably more complicated.

Chapter II: On the extension of Theorem II to a larger number of circular regions

5. **Problem of the locus corresponding to any number of circular regions.** Our attempt in Chapter I to extend Theorem II in a form to apply to the jacobian of two particular binary forms by considering regions other than circular as loci of the roots of the ground forms and finding the corresponding locus of the roots of the jacobian was not particularly fruitful. This seems to result rather from the nature of the problem itself than from the precise methods employed. We now take up the possibility of extending Theorem II so as to consider not merely three circular regions but any number of circular regions. Let us suppose explicitly that we have the binary forms f_1 and f_2 of respective degrees p_1 and p_2, and that the circular regions $C'_1, C'_2, \cdots, C'_m$ are the loci respectively of $p'_1, p'_2, \cdots, p'_m$ roots of f_1 and the circular regions $C''_1, C''_2, \cdots, C''_n$ are the loci respectively of $p''_1, p''_2, \cdots, p''_n$ roots of f_2, where we have

$$p'_1 + p'_2 + \cdots + p'_m = p_1,$$
$$p''_1 + p''_2 + \cdots + p''_n = p_2.$$

For convenience in phraseology, we shall suppose that none of these regions is either a point or the entire plane unless otherwise stated. We wish then to find the location of the roots of the jacobian of f_1 and f_2; not merely to determine certain regions in which lie or do not lie the roots of the jacobian, but to determine the actual *locus* of those roots under the assigned conditions, as in Theorem II.

Let us consider, for any particular values of the roots of f_1 and f_2 which satisfy our hypothesis, a root ζ of the jacobian exterior to all the circular regions $C'_1, \cdots, C'_m, C''_1, \cdots, C''_n$. This root ζ is an analytic function of α, any root of f_1 or f_2, and hence when α varies over a certain two-dimensional continuum, ζ also varies over a certain two-dimensional continuum. We thus have a certain number of regions which may or may not be distinct and may or may not have common points which are the loci of the points ζ. We see by the analyticity of the transformation that all the points α must be on the boundaries of their proper regions whenever a point ζ corresponding is on the boundary of its locus.* Moreover, if a point ζ is on the boundary of its locus, and exterior to all the regions $C'_1, \cdots, C'_m, C''_1, \cdots, C''_n$, we know by Lemma I (II, p. 102) that all the points α pertaining to any one circular region can be considered to coincide on the boundary of that region. But the precise manner of simultaneous variation of these coincident roots on the

* There is an exception to this reasoning if the algebraic equation defining ζ degenerates and if ζ is independent of a particular α, but in that case α can be moved at will without changing ζ and so α can be considered as on the boundary of its locus. A similar remark applies also below.

boundaries of their loci in such a manner that a point ζ or several points ζ remain on the boundaries of their loci and trace out those boundaries is as yet unknown.

Let us restrict ourselves for the moment to the situation where the circular regions $C_1', \cdots, C_n''$ are relatively small, or to be more precise, such that for no choice of the roots of the ground forms in their proper regions can two roots of the jacobian coalesce exterior to those circular regions; we suppose further that no two of the regions $C_1', \cdots, C_n''$ and the regions R which are the loci each of one of the roots of the jacobian exterior to those circular regions when the roots of the groundforms have their proper regions as loci—no two of all these regions have a point in common. We may allow the roots of the ground forms to coalesce in their proper regions; we notice that the circular regions $C_1', C_2', \cdots, C_m', C_1'', C_2'', \cdots, C_n''$ contain and are therefore the loci of respectively $p_1' - 1, p_2' - 1, \cdots, p_m' - 1, p_1'' - 1, p_2'' - 1, \cdots, p_n'' - 1$ roots of the jacobian. There are then $m + n - 2$ regions R each of which is the locus of one root of the jacobian. When we allow the circular regions to become larger and larger, of course the regions R expand also, need not preserve their identity (for example, two of them may coincide), and finally these regions cover the entire plane.

Very little is known of the precise nature of the boundaries of these regions R.* Their boundaries are not, except in very special instances, circular regions, but are curves which presumably have interesting properties with reference to the boundaries of the regions $C_1', \cdots, C_n''$; which properties can be expressed in a manner so as to be invariant under linear transformation. It is evidently true that if we start with any situation $C_1', \cdots, C_n''$ and if we allow two of the regions $C_1', \cdots, C_m'$ or two of the regions $C_1'', \cdots, C_n''$ to coalesce, one of the regions R will coalesce with them, and we shall have precisely the situation of $m - 1$ regions C_i' or $n - 1$ regions C_i''.

The exact determination of the regions R in any very general case seems difficult. If all the original circular regions reduce to points except one of them, say C_1', we can determine the path of the roots ζ of the jacobian as α, a p_1'-fold root of f_1, traces the circle C_1'. These roots ζ, in their totality, trace closed curves, for the situation when α returns to its initial position is exactly the same as the initial situation. The boundaries of the regions R must be composed of these closed curves, or at least of portions of them. If now we allow a second one of our circular regions, say C_1'', to be a non-degenerate region, the new locus of the roots of the jacobian will be a number of regions

* The writer conjectures that when there are q roots of the jacobian in these regions R, these regions are in their totality bounded by a degenerate or non-degenerate q-circular $2q$-ic; only the degenerate cases of this curve have ever been treated in detail, except for $q = 1$. Compare Walsh, Proceedings of the National Academy of Sciences, vol. 8 (1922), pp. 139–141.

R'. The boundaries of the regions R' will be curves which are envelopes of the curves R corresponding to the region C_1' and the null regions $C_2', \cdots, C_m', C_1'' = \beta, C_2'', \cdots, C_n''$, while the point β traces the circle C_1'''. By continuing in this way, we have a process for the generation of the regions R in any case desired. But the actual determination of the boundaries in a very general case would presumably be too laborious by this process; more powerful methods will have to be devised.

The statement has been made that the regions R are not in general circular regions; it is perhaps worth while to present a specific instance to illustrate this fact. We consider the polynomial

$$f(z) = (z - \alpha_1)(z - \alpha_2)(z - \alpha_3),$$

where $\alpha_1 = i, \alpha_2 = -i$, and the locus of α_3 is chosen to be a circle C_3 and its exterior, whose center is the origin and radius so large that the loci of the two roots of $f'(z)$ are entirely distinct. In the field of force to determine the roots of $f'(z)$, the force at a point of either coördinate axis due to the two particles at i and $-i$ is in direction along that axis. Hence, whenever $f'(z)$ has a root on an axis, α_3 must also be on that axis. The root ζ of $f'(z)$ larger in absolute value is determined on the positive half of the axis of reals by the particle α_3 at the right-hand intersection of C_3 and that axis, and a second particle of twice the mass at the harmonic conjugate* of ζ with respect to the points i and $-i$; this harmonic conjugate lies *to the left of* the origin. This root ζ is determined on the positive half of the axis of imaginaries by α at the upper intersection of C_3 and that axis, and a second particle of twice the mass at the harmonic conjugate of ζ with respect to i and $-i$; this harmonic conjugate lies *above* the origin. The curve bounding the locus of ζ is symmetric with respect to the coördinate axes and hence is not a circle.

The characteristic of Theorem II (and indeed also of Theorems VI and XI) in comparison with the more general results indicated in this present section seems to be a certain *linearity*. This fact is brought out very clearly in S, but also in II, since by Lemma II (II, p. 102) the position of equilibrium is determined by its cross ratio with three points, a relation which is essentially linear. It is as a result of that linearity that for our particular situations the loci of the roots of the jacobian are all bounded by circles.

6. **A condition that a root of the jacobian be on the boundary of its locus.** We return now to the general case of the preceding section, and shall obtain a geometric relation between the roots of the ground forms and a root of the jacobian, when all of those roots are on the boundaries of their loci.

Consider the points α_1 and α_2 at which coincide all the p_1' roots in C_1' and all the p_2' roots in C_2' respectively; we consider also a root ζ of the jacobian which

* Compare §§ 2, 9; also A, pp. 128–129.

is supposed not to lie at a common root of the two ground forms or at a multiple root of either form, so that ζ is given by a certain algebraic equation which actually contains α_1 and α_2. We may write this equation in the form

$$k + \frac{p_1'}{\zeta - \alpha_1} + \frac{p_2'}{\zeta - \alpha_2} = 0. \tag{1}$$

We can hold fast ζ and all roots of the ground forms other than α_1 and α_2, and move α_1 and α_2 depending on each other so as to satisfy (1). Since (1) is linear in α_1 and α_2, when one of these points is made to trace a circle the other also traces a circle. When α_1 moves so as to trace C_1', α_2 moves so as to trace a circle tangent to C_2'. In fact, if α_2 were to trace a circle intersecting C_2', α_2 would at some time move *interior* to the region C_2', and still α_1 and α_2 would be in their proper loci. Then motion of α_2 holding α_1 fast would cause ζ to move over a two-dimensional continuum, so ζ would not be on the boundary of its locus.

It will be useful to study in some detail the relation between α_1 and α_2 defined by (1). The two double points of the transformation (α_1, α_2) are

$$\alpha_1 = \alpha_2 = \zeta; \qquad \alpha_1 = \alpha_2 = \zeta + \frac{p_1' + p_2'}{k};$$

denote this latter point by α. We have, of course, $p_1' + p_2' \neq 0$, so that these two points are distinct. The cross ratio of these fixed points with α_1 and α_2 in any position is readily computed:

$$\frac{[\alpha_1 - \alpha_2]\left[\zeta - \left(\zeta + \frac{p_1' + p_2'}{k}\right)\right]}{[\alpha_2 - \zeta]\left[\left(\zeta + \frac{p_1' + p_2'}{k}\right) - \alpha_2\right]} = \frac{p_1' + p_2'}{p_2'}.$$

Inasmuch as this cross ratio is real, the four points α_1, α_2, ζ, α lie on a circle C.

The circle C is self-corresponding under the transformation (α_1, α_2). In fact, two of its points ζ and α are unchanged while a third point α_1 is transformed into a point α_2 of the circle; this is sufficient.

If p_1' and p_2' are both negative instead of both positive, we have the case of the roots of f_2 located in C_1'' and C_2''. In either of these situations, the first case we consider, the points α_1 and α_2 are separated by ζ and α. For a transformation can be made so that $k = 0$. The value of the cross ratio gives us

$$\frac{\alpha_1 - \alpha_2}{\zeta - \alpha_2} = \frac{p_1' + p_2'}{p_2'}, \qquad \frac{\alpha_1 - \zeta}{\zeta - \alpha_2} = \frac{p_1'}{p_2'};$$

so α_1 and α_2 are indeed separated by ζ and α.

We thus choose ζ as fixed on the boundary of its locus; the fixed points

α_1' and α_2' (particular values of α_1 and α_2 respectively) corresponding are also on the boundaries of their respective loci. We have already remarked that when α_1 moves from α_1' along C_1', α_2 moves from α_2' along a circle tangent to C_2'. When α_1 moves from α_1' interior to the region C_1', α_2 moves from α_2' exterior to the region C_2'. When α_1 moves on C, α_2 moves in the opposite direction but also on C. It follows that *C cuts C_1' and C_2' at angles of the same magnitude, and when C is transformed into a straight line the tangents to C_1' at α_1' and C_2' at α_2' are parallel.** There are different possibilities here according to whether C cuts C_1' and C_2' at equal angles or at supplementary angles. We leave it for the reader to notice that if the loci C_1' and C_2' are both interior or both exterior to their bounding circles, these two angles are equal; if one locus is interior and the other exterior to its bounding circle, these two angles are supplementary.

The second case we shall consider is that of two roots α_1 and α_2, of the forms f_1 and f_2, of multiplicities p_1' and p_1'', and loci C_1' and C_1'', respectively. Essentially the same formulas apply, except that in (1) and the succeeding formulas the numbers p_1' and p_2' are replaced by $p_2 p_1'$ and $-p_1 p_1''$ respectively; we suppose $p_2 p_1' - p_1 p_1'' \neq 0$. The cross ratio of the four points $\alpha_1, \alpha_2, \zeta, \alpha$ shows that α_1 and α_2 are not separated by ζ or α. Hence if α_1 and α_2 trace the self-corresponding circle C, they trace it in the same sense. The angles which C cuts on C_1' and C_1'' are then equal or supplementary according as the regions C_1' and C_1'' lie one inside and the other outside their bounding circles, or as these regions lie both inside or both outside their bounding circles. When C is transformed into a straight line, the lines tangent to C_1' at α_1 and to C_1'' at α_2 are parallel.

The third case we have to treat is the remaining situation under the second case, where $p_2 p_1' - p_1 p_1'' = 0$; here the transformation (α_1, α_2) has but the one double point ζ. It is still true that the circle C through $\alpha_1, \alpha_2, \zeta$ is self-corresponding under this transformation. For if we denote by γ the point α_2 corresponding to $\alpha_1 = \alpha_2'$, where α_1' and α_2' are fixed values of α_1 and α_2, we shall have

$$-k = \frac{p_2 p_1'}{\zeta - \alpha_1'} - \frac{p_1 p_1''}{\zeta - \alpha_2'} = \frac{p_2 p_1'}{\zeta - \alpha_2'} - \frac{p_1 p_1''}{\zeta - \gamma},$$

$$\frac{(\alpha_1' - \alpha_2')(\gamma - \zeta)}{(\alpha_2' - \gamma)(\zeta - \alpha_1')} = -1,$$

so $\alpha_1', \alpha_2', \zeta, \gamma$ are concyclic; the three points $\alpha_1', \alpha_2', \zeta$ of C are transformed

* We cannot prove here, as in III, Theorem II, that this property holds also for the tangent to the boundary of the locus of ζ at the point ζ. In fact, if we choose another pair of points α_1', α_2', leading to the circle C'', it is in general impossible for C'' to cut at equal angles C_1' at α_1' and the boundary of the locus of ζ at ζ. For a specific example, see the illustration used at the close of § 5.

into three points α_2', γ, ζ of C which, therefore, is self-corresponding. If ζ is transformed to infinity, we have

$$\frac{\alpha_1' - \alpha_2'}{\alpha_2' - \gamma} = 1,$$

so γ is obtained from α_2' by translation by an amount equal to $\alpha_1' - \alpha_2'$. Then α_1 and α_2 trace C in the same sense. As in our second case, the angles which C cuts on C_1' and C_1'' are equal or supplementary according as the regions C_1' and C_1'' lie one inside and the other outside their bounding circles, or as these regions lie both inside or both outside their bounding circles. When C is transformed into a straight line, the lines tangent to C_1' at α_1 and to C_1'' at α_2 are parallel.*

We have now considered all typical cases of two roots α_1 and α_2 of the ground forms. In particular if we choose two roots of a single form which have the same locus, the reasoning we have used shows that when ζ is on the boundary of its locus α_1 and α_2 must be on the boundary of their common locus and *must coincide*. This may be used to replace Lemma I (II, p. 102). The reader may be interested in applying the remarks of the present section to the situations of Theorems II and VI.

7. **A special case of coaxial circles.** We have pointed out in § 5 that for the situation there described of $m + n$ circular regions $C_1', \cdots, C_m', C_1'', \cdots, C_n''$ as the loci of the roots of the ground forms, the locus of the roots of the jacobian is not in general a number of circular regions or of regions bounded by several circles. But of course there are special situations for which the locus of the roots of the jacobian is bounded by circles. This is evidently true, for example, if $C_1', \cdots, C_m', C_1'', \cdots, C_n''$ are bounded by $m + n$ concentric circles or more generally by $m + n$ coaxial circles having no common point.† For such a situation, moreover, the methods used in I can be applied; we shall give an illustration of that fact.

We use the notation of I, p. 293 ff., and suppose the situation simplified

* This remark enables us immediately to state something about the locus to be determined in certain cases. Thus consider the situation and notation of Theorem X (we might indeed choose that of Theorem VI). Choose the line through $\alpha_1, \cdots, \alpha_n$ horizontal and draw a parallel L' through any point z on the boundary of its locus. If one point ζ_1 (notation of § 9) is not on L', say below L', and if z is exterior to all the circles $C_1, \cdots, C_n$, all other points ζ_i are also below L'. Then by Lucas's Theorem, z cannot be a root of $f'(z)$. Therefore all points ζ_i lie on L' and z lies on a circle C_i'.

If $m_1, \cdots, m_n$ are all greater than unity, no point z interior to a circle C_i need be considered. If any m_i is unity, however, the points z interior to C_i must be considered. The writer has been unable to treat similarly (by the method just indicated) this last case, and thus completely to prove Theorem X.

† If the coaxial system is composed of all circles through two points or all circles tangent at a single point, we may consider all the roots of both forms to coincide at a single point, the jacobian vanishes identically, and the locus of the roots of the jacobian is the entire plane.

by transformation as in I. Suppose C_1 to contain k roots of f_1 ($p_2 k$ positive particles) and l roots of f_2 ($p_1 l$ negative particles). If l is sufficiently small in comparison with k, and if C_2 and C_3 are sufficiently remote, it seems reasonable to suppose that we can obtain a region near but exterior to C_1, which region contains no root of the jacobian. The circle C_1 contains, then, k particles each of mass p_2 and l particles each of mass $-p_1$. Outside of C_2 there are $p_1 - k$ particles each of mass p_2 and outside of C_3 there are $p_2 - l$ particles each of mass $-p_1$. If Q is a position of equilibrium, we must have, in the notation of I,

$$\frac{p_2 k}{a+r} \leqq \frac{p_1 l}{r-a} + \frac{p_2(p_1-k)}{b-r} + \frac{p_1(p_2-l)}{c+r},$$

which can be put into the equivalent form

$$\begin{aligned} 0 \leqq\ & r^2[-p_2 k(a+b) - p_1 l(a+c) + p_1 p_2(b+c)] \\ & + r[p_2 k(a+b)(a-c) + p_1 l(a+c)(b-a)] \\ & + [-p_1 p_2 a^2(b+c) + p_2 kac(a+b) + p_1 lab(a+c)]. \end{aligned} \tag{2}$$

This form does not simplify materially. Denote by C_4 and C_5 the circles whose centers are O and radii the roots of the right-hand member. The cross ratio of the points C_4', C_4'', C_5', C_5'' (notation as in I, p. 294) with the collinear points C_1'', C_2', C_3'' can easily be calculated, but this cross ratio contains a, b, c explicitly and is not independent of their ratios; we therefore use a different method to describe C_4 and C_5. We are supposing implicitly that the roots of the right-hand member of (2) are positive or that at least one of these roots is positive.

If C_4 and C_5 lie between C_1 and C_2 and between C_1 and C_3, they bound an annulus which contains no root of the jacobian. For if $r = a$, the right-hand member of (2) reduces to

$$2p_1 l(a+c)(b-a),$$

so that inequality is satisfied for $r = a$ and therefore is not satisfied when r lies between the two roots.

Under this hypothesis we can determine the precise number of roots of the jacobian in the smaller of the new circles by allowing the roots of f_1 and f_2 in C_1 to move continuously and to coincide at O. When the $p_2 k$ and $p_1 l$ particles are all in coincidence at O, the circle C_1 contains precisely $k + l - 1$ roots of the jacobian, so this is the original number of roots interior to or on the inner boundary of the annulus.

Hence we have, under the assumptions already made:

1. *If the circles C_4 and C_5 lie between C_1 and C_3, then the annular region between C_4 and C_5 contains no root of the jacobian of f_1 and f_2. The region which*

is bounded by C_4 or C_5 and contains the region C_1 contains precisely $k+l-1$ roots of the jacobian.

2. *If C_4 and C_5 are separated by C_3, there are no roots of the jacobian in the annular region which is part of the annular region bounded by C_4 and C_5 and which contains no point of the region C_3. The circular region bounded by C_4 or C_5 which contains the region C_1 but no point of the region C_3 contains precisely $k+l-1$ roots of the jacobian.*

This theorem can readily be expressed in general form so as to include the situation after linear transformation; compare the corresponding statement in I.

8. **Theorem VI, a general theorem for circles having an external center of similitude.** There is another fairly general class of loci other than the very simple class just considered for which the locus of the roots of the jacobian as treated in § 5 is bounded by circles. We shall now use a method which is novel in some respects but which makes use of our former results to establish

THEOREM VI. *Let the interiors and boundaries of the circles $C'_1, C'_2, \cdots, C'_{n'}$, whose centers are $\alpha'_1, \alpha'_2, \cdots, \alpha'_{n'}$, respectively, be the loci of $m'_1, m'_2, \cdots, m'_{n'}$ roots of the form f_1 which has no other roots. Let the interiors and boundaries of the circles $C''_1, C''_2, \cdots, C''_{n''}$, whose centers are $\alpha''_1, \alpha''_2, \cdots, \alpha''_{n''}$, respectively, be the loci of $m''_1, m''_2, \cdots, m''_{n''}$ roots of the form f_2 which has no other roots. Suppose further that a point P is interior to no circle C'_i or C''_i and is an external center of similitude for every pair of the circles C'_i, C'_j and for every pair of the circles C''_i, C''_j and an internal center of similitude for every pair C'_i, C''_j. The distinct roots of $\bar{J}$, the jacobian of $\bar{f}_1$ and $\bar{f}_2$, when all the roots of the ground forms are concentrated at the centers of their proper circles, are denoted by $\alpha_1, \alpha_2, \cdots, \alpha_n$ (all of which points are collinear with $P, \alpha'_1, \cdots, \alpha'_{n'}, \alpha''_1, \cdots, \alpha''_{n''}$) of multiplicities $m_1, m_2, \cdots, m_n$, and by $C_1, C_2, \cdots, C_n$ are denoted the circles which have these points as centers and radii such that P is an external or internal center of similitude for every pair of the circles $C'_1, \cdots, C'_{n'}, C''_1, \cdots, C''_{n''}, C_1, \cdots, C_n$. Then the locus of the roots of J, the jacobian of f_1 and f_2, is composed of the interiors and boundaries of the circles $C_1, C_2, \cdots, C_n$. A circle C_i exterior to all the other circles C_j contains precisely m_i roots of J.*

Limiting cases of the circles $C'_1, \cdots, C'_{n'}, C''_1, \cdots, C''_{n''}$ are the points P and P', the point at infinity. We shall admit these circles as possibilities in the demonstration of the theorem, providing, however, that there is at least one proper circle, which we shall suppose to be C'_1. If there is no proper circle C'_1 or C''_1, either the only roots of J are P and P', in which case the theorem remains true, or every point of the plane is a root of J, in which case the theorem is not true.

The configuration of the three sets of circles has some obvious but interesting properties relative to $\bar{J}$. Let us choose as horizontal the line L on which lie

$P, \alpha_1', \cdots, \alpha_{n'}', \alpha_1'', \cdots, \alpha_{n''}''$, with the C_i' to the right of P, let us number in order the two sets of circles commencing with C_1' and C_1'', the nearest circles to P, and let us denote by μ_i' the left-hand intersection of C_i' with L and by ν_i' the right-hand intersection, with the opposite conventions for μ_i'' and ν_i'', the intersections of C_i'' with L. The points $\mu_1', \cdots, \mu_{n'}', \mu_1'', \cdots, \mu_{n''}''$ may be obtained from the points $\alpha_1', \cdots, \alpha_{n'}', \alpha_1'', \cdots, \alpha_{n''}''$ by a similarity transformation with P as center, and as a line L' is allowed to rotate about P its intersections with C_i' and C_i'' have always this same property. In fact, we may consider properly chosen intersections of L with C_i' and C_i'' to be the roots of $\bar{f}_1$ and $\bar{f}_2$; then the roots of $\bar{J}$ trace the circles C_i. In particular, when L' is tangent to the circles $C_1', \cdots, C_{n'}', C_1'', \cdots, C_{n''}''$, the points of tangency have this relation to the points of tangency with $C_1, \cdots, C_n$.

To prove Theorem VI, we consider as usual the field of force given by Theorem I (I, p. 291 = I, p. 101). We can obtain immediately a qualitative idea of the locus of the roots of J. No point Q above both of the tangents T and T' common to the circles $C_1, \cdots, C_n, C_1', \cdots, C_{n'}', C_1'', \cdots, C_{n''}''$ can be a root of J. For the force at Q due to the particles situated at the roots of f_1 has a component normal to PQ and that due to the particles situated at the roots of f_2 has a component normal to PQ and in the same sense; at least one of these components is different from zero. Thus also no point below both T and T' can be a root of J and no point above T (or T') but not above T' (or T) yet lying to the right of C_1'' and to the left of C_1' can be a root of J. No point Q of T (or T') can be a root of J unless all the roots of f_1 and of f_2 are on T (or T') which can only occur if these roots lie at the points of tangency of T (or T') with the circles bounding their proper loci; that is, if Q lies at a point of tangency of T (or T') and a C_i. Inasmuch as the locus of the roots of J is a closed point set there must be some sort of a boundary of that locus between any two of the circles C_i.

By means of the similarity transformation with P as center, we see that every point of the locus as stated in Theorem VI is really a point of the locus. To complete the determination of the locus we have merely to prove that if a point Q is a point of the boundary of the locus, that point is on one of the circles C_i.

The interior and boundary of the circle C_i' or C_i'' which is the locus of more than one root of f_1 or of f_2 is also the interior and boundary of one of the circles C_i; every point interior to or on such a circle is a point of the locus of the roots of J. A point on or interior to two circles C_i' and C_{i+1}' (and so of course C_i'' and C_{i+1}'') is also on or within a circle C_j; in fact there is a root α_j of $\bar{J}$ between α_i' and α_{i+1}', and the circle C_j whose center is α_j contains in its interior the region common to C_i' and C_{i+1}', for this is true of any circle whose center lies between α_i' and α_{i+1}' if the circle has the two common tangents T and T'.

It merely remains to consider points Q exterior to all the circles $C_1', \cdots, C_{n'}', C_1'', \cdots, C_{n''}''$ or interior to or on at most one circle C_i' or C_i'' which is the locus of but one root of f_1 or f_2. In any case the particles at the roots of f_1 and f_2 may by Lemma I (II, p. 102) be considered to coincide in their respective loci so far as the force at Q is concerned.

9. **Proof of Theorem VI; replacing of two particles by a single particle.** The point Q is then to be considered as fixed, and for definiteness to lie to the right of P and of course above one of the lines T and T' but below the other, so that the line PQ actually cuts all the circles $C_1', \cdots, C_{n'}', C_1'', \cdots, C_{n''}''$. We know the loci of certain particles each representing several of the roots of f_1; we shall replace two of these particles by a single equivalent particle, then the new particle and a third of the original particles by a single equivalent particle, and so on until we have replaced all the particles representing the roots of f_1 by a single equivalent particle, and similarly for the particles representing the roots of f_2. A study of the properties of the loci of these various particles will enable us to prove that Q is on one of the circles C_i. We suppose for the present that Q is exterior to all the circles C_j'.

To replace two particles at ζ_1 and ζ_2 of masses m_1 and m_2 by a single particle ζ of mass $m_1 + m_2$ so that the force at Q shall be unchanged, we have the equation for ζ

$$\frac{m_1}{\zeta_1 - q} + \frac{m_2}{\zeta_2 - q} = \frac{m_1 + m_2}{\zeta - q},$$

where q is the complex number representing the point Q. This equation is equivalent to

$$\frac{(\zeta_1 - q)(\zeta_2 - \zeta)}{(\zeta_2 - q)(\zeta - \zeta_1)} = \frac{m_1}{m_2}.$$

We wish to replace the particles ζ_1 and ζ_2, whose loci are the interiors and boundaries of C_1' and C_2' and which represent all the roots of f_1 having these regions as loci, by a single equivalent particle. If we transform Q to infinity, we shall have precisely the conditions of III, Theorem VIII; m_1 and m_2 are both positive and the points ζ_1 and ζ_2 are always separated by ζ and Q. Then the variable circle C of Lemma IV (II, p. 105) moves so as always to cut C_1' and C_2' at the same angle, and cuts also S_1', the circle bounding the locus of ζ, also at this same angle. When Q is transformed back to the finite part of the plane, it remains true that C cuts C_1' and C_2' at the same angle; C cuts S_1' at this same angle or the supplementary angle according as the locus S_1' is interior or exterior to its bounding circle.* We leave this fact to be verified by the reader; this can be done by considering any one circle C under the

* This difference in behavior, which we shall constantly meet, disappears entirely if we project stereographically on to the sphere.

transformation of Q from the point at infinity to its original finite position. In particular it will be noticed that the line PQ is one of the circles C cutting C'_1, C'_2, S'_1 at the proper angles. When ζ_1 and ζ_2 are on PQ and are chosen as the right-hand (left-hand) intersections of PQ with C'_1 and C'_2, ζ is on PQ and at the right-hand or left-hand (left-hand or right-hand) intersection of PQ with S'_1 according as the locus S'_1 is interior or exterior to its bounding circle. The converse of this statement is also true; such a choice of ζ leads to a unique determination of ζ_1 and ζ_2 as described.

We have supposed Q to be exterior to all the circles C'_i; suppose now Q exterior to C'_1 but interior to C'_2; we need not consider Q interior to the two circles. When Q is transformed to infinity we have a special case of Theorem I, but no longer a special case of III, Theorem VIII. The circle C which generates as in Lemma IV (II, p. 105) the boundary of S'_1 cuts C'_1 and C'_2 at supplementary angles. In fact, if we assume C to cut C'_1 and C'_2 at equal angles, but not at supplementary angles, when ζ_1, ζ_2, ζ are on the boundaries of their loci, the line $\zeta_1\,\zeta_2\,\zeta$ can be rotated about ζ so that ζ_1 and ζ_2 move into the interiors of their loci, so ζ cannot be on the boundary of its locus. The circle S'_1 is found to be cut by C at an angle equal to that cut on C'_2 and supplementary to that cut on C'_1.

We shall not consider in detail the case that Q is on C'_1 or C'_2; we need not consider Q on both circles. Whether Q is on or within one circle or exterior to all the circles C'_i it is always true that when Q is in its original finite position C cuts C'_1 and C'_2 at the same angle, and cuts S'_1 at this same angle or the supplementary angle according as the locus S'_1 is interior or exterior to its bounding circle. When ζ_1 and ζ_2 are chosen properly as the intersections of C'_1 and C'_2 with PQ, one of the circles C, ζ is on PQ and on S'_1, and conversely. The tangents to these three circles at those three points are parallel.

We have thus replaced the particles at ζ_1 and ζ_2 by a single equivalent particle. So far as the force at Q is concerned, we can replace ζ_1 and ζ_2 at any positions in their loci by ζ in its locus S'_1, and for any position of ζ in S'_1 we can determine ζ_1 and ζ_2 in their loci so that the force at Q is the same. If Q is in C'_1 or C'_2, the force at Q can be made as large as desired, so Q must be in S'_1, and conversely. If Q is on C'_1 or C'_2, the force at Q can be made as large as desired but only in certain special directions, so Q is on S'_1, and conversely. If the region S'_1 is external to its bounding circle, the force at Q is zero for proper choice of ζ and hence of ζ_1 and ζ_2, and conversely.

Next we replace by a single equivalent particle ζ' the particle ζ as just determined and ζ_3, the particle which represents the roots of f_1 whose common locus is C'_3 (assuming the existence of this set of roots). No further detailed discussion is required of this new situation; as before, a system of circles C' cuts S'_1, C'_3, and the boundary S'_2 of the locus of ζ' at equal angles or at angles

supplementary to the angle cut on C_3' according as the loci S_1' and S_2' do not or do include the point at infinity. The line PQ is as before one of the system of circles C'. When ζ' is on PQ and on S_2', ζ_3 is on PQ and on C_3' and ζ is on PQ and S_1', so ζ_1 and ζ_2 are on PQ and C_1' and C_2' respectively; moreover, the tangents to C_1', C_2', C_3' at ζ_1, ζ_2, ζ_3, respectively, are parallel. The converse of this statement is also true.

We continue in this same manner to replace pairs of particles by equivalent particles, and finally replace all the particles $\zeta_1, \zeta_2, \cdots, \zeta_{n'}$ representing the roots of f_1 by a single particle η_1 whose locus is a circular region S_1 whose boundary is cut by PQ at an angle supplementary or equal to the angle cut on C_1', C_2', $\cdots$, $C_{n'}'$ according as the locus contains or does not contain the point at infinity. When η_1 is on PQ and on S_1, we know that $\zeta_1, \cdots, \zeta_{n'}$ are on $C_1', \cdots, C_{n'}'$ respectively and that the tangents to these circles at these points are parallel. It follows from reasoning to be given later that at no intermediate stage is the locus of one of our auxiliary particles the entire plane.

Similarly the particles ξ_1, ξ_2, $\cdots$, $\xi_{n''}$ representing the roots of f_2 are replaced by a single particle η_2 whose locus is a circular region S_2 which is either one of the points P or P' or is bounded by a circle S_2 which is cut by PQ at an angle equal to the angles cut on the circles C_1'', C_2'', $\cdots$, $C_{n''}''$. In fact, if all the roots of f_2 are not concentrated at P' the particles corresponding to the roots of f_2 always exert at Q a force not zero, so the locus of η_2 does not include the point at infinity. When η_2 is on PQ and S_2, we know that $\xi_1, \xi_2, \cdots, \xi_{n''}$ are on C_1'', C_2'', $\cdots$, $C_{n''}''$ respectively, and that the tangents to these circles at these points are parallel.

10. **Theorem VI: proof completed.** For Q to be a root of J, the loci S_1 and S_2 must have at least one point in common, at which point are to coincide η_1 and η_2 so that their resultant force at Q shall be zero. Such a common point cannot be Q, for Q is not a point of S_2. The loci S_1 and S_2 cannot overlap if Q is on the boundary of its locus, for when Q varies slightly in any direction, S_1 and S_2 vary but slightly. If S_1 and S_2 overlap, we may vary Q slightly in any direction but so little that S_1 and S_2 still have common points, so that Q remains a root of J for some choice of η_1 and η_2 and hence is not on the boundary of the locus of the roots of J. We defer until later the possibilities that the two circles S_1 and S_2 coincide or that S_1 or S_2 may be the entire plane or a point.

The regions S_1 and S_2 have but a single point in common, and since PQ cuts the circles S_1 and S_2 at equal or supplementary angles according as S_1 does or does not contain the point at infinity, a single point common to these two loci must lie on PQ. That is, η_1 and η_2 lie on S_1 and S_2 respectively, and on PQ, so $\zeta_1, \zeta_2, \cdots, \zeta_{n'}, \xi_1, \xi_2, \cdots, \xi_{n''}$ all lie on PQ and the lines tangent

to the circles $C'_1, C'_2, \cdots, C'_{n'}, C''_1, C''_2, \cdots, C''_{n''}$ at these points are parallel. Then Q lies on one of the circles C_i.*

The possibility that the circles S_1 and S_2 coincide is readily treated. We may choose η_1 and η_2 to coincide on S_1 and on S_2, and on PQ. These points are still on the boundaries of their respective loci, and hence the previous reasoning is valid.

According to the assumptions already made, the locus S_1 cannot be a point. The locus S_2 will be a point when and only when the roots of f_2 are concentrated at P or P' or both. But in such a case the single point S_2 is on the line PQ and the preceding reasoning holds.

The possibility that S_1 or S_2 may be the entire plane remains to be considered. If one of these loci is the entire plane, the other must be a point; otherwise we have essentially the case of overlapping already disposed of. The locus S_2 is either the point P' or does not contain P', so is never the entire plane. If S_1 is the entire plane, we may suppose S_2 to be a point which of course lies on PQ. We prove our former result by a limiting process. When a point Q' is very near Q but external to the locus of the roots of J, the circles $S'_1, S'_2, \cdots, S_1$ are very near the corresponding circles for Q; for Q' the locus S_1 is certainly not the entire plane. Denote by Σ_2 the point at which is located the single particle representing all the roots of f_2, so far as concerns the force at Q'; Σ_2 is not in the locus S_1. When Q' approaches Q always remaining exterior to the locus of the roots of J, Σ_2 approaches the point S_2. The circle S_1 corresponding to Q' becomes smaller and smaller, the locus S_1 never contains Σ_2, so the circle S_1 approaches the point S_2. We may choose η_1 an intersection with PQ' of the circle S_1 corresponding to Q', and we shall have the points $\zeta_1, \cdots, \zeta_{n'}$ on PQ' and on the circles $C'_1, \cdots, C'_{n'}$. When Q' approaches Q, PQ' approaches PQ, the point η_1 approaches S_2 and the points $\zeta_1, \cdots, \zeta_{n'}$ approach points on PQ and on $C'_1, \cdots, C'_{n'}$. These limiting points can be taken as corresponding to η_1 coinciding with S_2 and thus give our result that Q lies on one of the circles $C_1, \cdots, C_n$.

Theorem VI is now completely proved except for its last sentence. When we notice the number of roots of J in a region C_i and remark that if the roots of f_1 and f_2 are varied continuously then the roots of J vary continuously, and that if C_i is exterior to every other circle C_j no root can enter or leave C_i, this last sentence is seen to be true. It hardly need be added that a number of circles C_i which may have common points but which have no point in common with any other circle C_j contain a number of roots of J equal to the sum of the multiplicities m_i corresponding to their centers as roots of J.

* The mere fact that the tangents at these points are parallel does not rule out certain isolated points Q on the line $P\alpha'_1 \cdots \alpha'_{n'} \alpha''_1 \cdots \alpha''_{n''}$ but a more detailed consideration of the loci $S'_1, S'_2, \cdots$ does rule them out without difficulty.

11. **Generalization of Theorem VI by transformation.** In Theorem VI we have assumed that P is interior to none of the circles $C_1', \cdots, C_{n'}', C_1'', \cdots, C_{n''}''$. The theorem is still true if this assumption is omitted, even if we permit roots of one or both forms to lie at infinity, except that the locus of the roots of J may be the entire plane, and will surely be the entire plane if P is interior to or on the circles $C_1', \cdots, C_{n'}', C_1'', \cdots, C_{n''}''$, and if no roots of either ground form are constrained to lie at P'. The proof as given requires only a few minor modifications to apply to this new configuration. If the circles $C_1', \cdots, C_{n'}', C_1'', \cdots, C_{n''}''$ all have the common center P, the circles bounding the locus of the roots of J are not determined by the position of their centers as described in the statement of Theorem VI, but are to be determined for example by their points of intersection with an arbitrary line through P, precisely as we considered the points μ_i' and μ_i'' in § 8.

Theorem VI has the advantage over Theorem II of being entirely symmetrical with respect to the two forms f_1 and f_2. The special case of Theorem VI where there are two circles C_1', C_2' and merely one circle C_1'' leads to merely one circle C_1 distinct from the three original circles. For this case, Theorems II and VI give the same result. But of course Theorem II is more general than this particular situation. Thus, the result for the jacobian problem of the theorem stated in II, pp. 114–115, or indeed of the problem of § 9 where Q is interior to C_2' is included in Theorem II but not in Theorem VI. There is, however, a general theorem concerned with an indefinite number of circular regions $C_1', \cdots, C_{n'}', C_1'', \cdots, C_{n''}''$ which generalizes all possible situations of Theorem II and which is to be proved in §§ 13–15 by the methods we have been using.

Theorem VI as stated is not invariant under linear transformation. If we perform such a transformation we obtain the following new result:

THEOREM VII. *If the loci of the roots of f_1 are circular regions bounded by circles each of which is tangent internally to a circle L_1 and externally to a circle L_2 (tangent to both L_1 and L_2 internally) and if the loci of the roots of f_2 are circular regions bounded by circles each of which is tangent externally to L_1 and internally to L_2 (tangent to both L_1 and L_2 externally), and if these loci are so related to their bounding circles that they contain neither the entire circle L_1 nor the entire circle L_2, then the locus of the roots of the jacobian J of f_1 and f_2 is a number of circular regions bounded by circles each tangent internally to L_1 and externally to L_2 or tangent externally to L_1 and internally to L_2 (tangent to L_1 and L_2 internally or to L_1 and L_2 externally) and such that each region is so related to its bounding circle that it contains neither the entire circle L_1 nor the entire circle L_2. The exact location of the circles bounding the locus of the roots of J can be determined by allowing the roots of the ground forms to coincide on L_1 or on L_2, always remaining in their proper loci; the roots of J are the points of tangency with L_1*

and L_2 *of the circles desired. The number of roots of* J *in any of these circular regions having no point in common with any other of these regions is the multiplicity as a root of* J *of the points of tangency of its boundary with* L_1 *and* L_2 *under these conditions.*

There is a limiting case of this theorem yet different from Theorem VI where L_1 is a straight line and L_2 a proper circle, but we shall not state the result in detail.

12. **Application of Theorem VI to the roots of the derivatives of polynomials.** Theorem VI and in fact Theorem VII can be used to obtain results concerning the roots of the derivative of a rational function by means of the remark made in I, p. 297 (or II, p. 114). We thus consider $C_1', \cdots, C_{n'}'$ as the loci of the roots and $C_1'', \cdots, C_{n''}''$ as the loci of the poles of the given function. The regions $C_1, \cdots, C_n$ together with the possibility of the point at infinity appear as the loci of the roots of the derivative. There is a peculiar difference, however, between this locus and the corresponding locus of the roots of the jacobian. A region C_i'' which is the locus of more than one pole of the original function is the locus of at least one root of the derivative, with the exception that no point of the bounding circle C_i'' can be a root of the derivative unless it is interior to another region C_j''; we leave to the reader the verification of this statement.

We shall dwell at some length on an application of the above remark applied to Theorem VI, which is indeed a special case of that theorem, concerning the derivative of a polynomial:

THEOREM VIII. *Let the interiors and boundaries of circles* $C_1, \cdots, C_n$ *whose centers are* $\alpha_1, \cdots, \alpha_n$ *be the loci of* $m_1, \cdots, m_n$ *roots respectively of a polynomial* $f(z)$ *which has no other roots; suppose these circles to have a common external center of similitude* P *actually exterior to all these circles. Denote by* $g(z)$ *the polynomial* $f(z)$ *when all its roots are concentrated at the centers of their proper circles, and denote by* $\alpha_1', \cdots, \alpha_{n'}'$ *the distinct roots of its derivative* $g'(z)$, *of respective multiplicities* $m_1', \cdots, m_{n'}'$. *Then the locus of the roots of* $f'(z)$ *is composed of the interiors and boundaries of the circles* $C_1', \cdots, C_{n'}'$ *whose centers are* $\alpha_1', \cdots, \alpha_{n'}'$ *and whose radii are such that* P *is a common external center of similitude for the circles* $C_1, \cdots, C_n, C_1', \cdots, C_{n'}'$. *A circle* C_i' *which has no point in common with another circle* C_j' *contains* m_i' *roots of* $f'(z)$.

An extreme degenerate case of this theorem is when all the C_i are null circles, and $f(z)$ is identical with $g(z)$. The case of merely two circles brings us back to a theorem given in II, p. 115.

Inasmuch as the circles C_i' have P as a common external center of similitude, Theorem VIII can be applied again to the polynomial $f'(z)$ and shows that the roots of $f''(z)$ lie on or within certain circles $C_1'', \cdots, C_{n''}''$. The most obvious consideration of the geometric situation shows that any point on or within one of these circles actually is a point of the locus.

It is to be noticed, however, that this reasoning can be used only if we know that the circles C'_i contain respectively m'_i roots of $f'(z)$, and it is pointed out in § 4 that one of these circles does not necessarily contain precisely that number of roots of $f'(z)$ and in fact may contain no such root. Thus the reasoning can be used only if we know that the circles C'_i are mutually external. This is always the case for a given set of values of $\alpha_1, \cdots, \alpha_n$ if the circles C_i are sufficiently small, so in the following theorem we require that the circles C_i be sufficiently small. This means, more explicitly, that the theorem is true for a definite derivative $g^{(k)}_{(z)}$ if the circles C_i are so small that no circle $C^{(m)}_i$ has a point in common with a circle $C^{(m)}_j$ $(i \neq j)$, for $m = 1, 2, \cdots, k-1$.

Thus we have

THEOREM IX. *Let the interiors and boundaries of circles $C_1, \cdots, C_n$ whose centers are $\alpha_1, \cdots, \alpha_n$ be the loci of $m_1, \cdots, m_n$ roots respectively of a polynomial $f(z)$ which has no other roots; suppose these circles to have a common external center of similitude P actually exterior to all these circles. Denote by $g(z)$ the polynomial $f(z)$ when all its roots are concentrated at the centers of their proper circles, and denote by $\alpha^{(k)}_1, \cdots, \alpha^{(k)}_{n(k)}$ the distinct roots of its kth derivative $g^{(k)}(z)$, of respective multiplicities $m^{(k)}_1, \cdots, m^{(k)}_{n(k)}$. Then if the circles C_i are sufficiently small the locus of the roots of $f^{(k)}(z)$ is composed of the interiors and boundaries of the circles $C^{(k)}_1, \cdots, C^{(k)}_{n(k)}$ whose centers are $\alpha^{(k)}_1, \cdots, \alpha^{(k)}_{n(k)}$, and whose radii are such that P is a common external center of similitude for the circles $C_1, \cdots, C_n, C^{(k)}_1, \cdots, C^{(k)}_{n(k)}$. A circle $C^{(k)}_i$ which has a point in common with no other circle $C^{(k)}_j$ contains precisely $m^{(k)}_i$ roots of $f^{(k)}(z)$.*

The special case of this theorem where there are but two of the original circles C_1 and C_2 has already been proved by another method.* For this special case we make no restriction on the size of the circles C_i.

A limiting case of Theorem IX is that P is infinite but the points $\alpha_1, \cdots, \alpha_n$ finite, and the radii of $C_1, \cdots, C_n$ finite. The circles $C_1, \cdots, C_n$ are then all equal. The theorem is true for this limiting case. In fact, suppose a root R of $f^{(k)}(z)$ to be exterior to all the circles $C^{(k)}_1, \cdots, C^{(k)}_{n(k)}$. We can choose circles $S_1, \cdots, S_n$ having a finite point P as common external center of similitude and such that R is also exterior to all the circles $S^{(k)}_1, \cdots, S^{(k)}_{n(k)}$ corresponding. This shows that every point of the locus is on or within $C^{(k)}_1, \cdots, C^{(k)}_{n(k)}$; the converse is easily seen from translation of the situation for $g(z)$ and $g^{(k)}(z)$. This result may be expressed somewhat loosely as follows:

THEOREM X. *If the loci of the roots of a polynomial are the interiors and boundaries of sufficiently small equal circles whose centers lie on a line L, the locus of the roots of the kth derivative $f^{(k)}(z)$ consists of the interiors and bound-*

* Walsh, Paris Comptes Rendus, vol. 172 (1921), pp. 662–664.

aries of circles equal to these whose centers also lie on L and depend only on the centers of the original circles.

This new theorem for $k = 1$ is not a special case of Theorem VI and can easily be expressed in a form invariant under linear transformation, thus giving a new result for the jacobian of two binary forms (compare § 14) and for the derivative of a rational function.*

The approximate determination of the roots of the jacobian of two binary forms, of the derivative of a rational function, or of any derivative of a polynomial is thus made, by Theorems VI–X, to depend essentially on the determination of the roots of the jacobian, of the derivative of a rational function, or of any derivative of a polynomial which has all its roots real.

The extreme simplicity of Theorem X immediately raises the question of the truth of that theorem if the supposition of the collinearity of $\alpha_1, \cdots, \alpha_n$ is omitted. We can easily prove that the theorem is not true under this changed hypothesis by means of the remark of § 6. It is surely true under the changed hypothesis that every point on or within a circle C'_i that is equal to the C_i and whose center is α'_i is a point of the locus, but it is not true without further restrictions that the locus consists precisely of the points on and interior to the circles C'_i.

Consider a polynomial $g(z)$ with three simple roots $\alpha_1, \alpha_2, \alpha_3$, which are not collinear, so that neither root α'_1, α'_2 of $g'(z)$ is collinear with a pair of the points $\alpha_1, \alpha_2, \alpha_3$. Choose the equal circles C_1, C_2, C_3, with centers $\alpha_1, \alpha_2, \alpha_3$, of such small radius that for no possible choice of the points in their proper loci can we have two roots β_i and β_j of $f(z)$ collinear with a root β'_k of $f'(z)$. Suppose β'_1 to be on C'_1; we choose β_i of such a nature that $\beta_i - \alpha_i = \beta'_1 - \alpha'_1$. The circle C through $\beta_1, \beta_2, \beta'_1$ is not a straight line, the points β_1 and β_2 cannot satisfy the requirements of § 6 with regard to the circles C, C_1, and C_2, from which it follows that β'_1 cannot be on the boundary of its locus.

13. **Theorem XI: an extension of Theorems II and VI.** We now come to the proof of the general theorem mentioned in § 11, which includes Theorem II as well as Theorem VI:

THEOREM XI. *Let f_1 and f_2 be two binary forms, and let circular regions $C'_1, C'_2, \cdots, C'_{n'}$; $C''_1, C''_2, \cdots, C''_{n''}$ be the respective loci of $m'_1, m'_2, \cdots, m'_{n'}$ roots of f_1 (which has no other roots) and of $m''_1, m''_2, \cdots, m''_{n''}$ roots of f_2 (which has no other roots). Suppose there is a family of coaxial circles S each of which cuts at the same angle all the circles C'_i which bound loci interior to them, and at*

* (Added in proof): It seems to the writer probable that Theorem IX is true with no restriction on the size of the circles C_i. The special case of this more general theorem where the circular regions C_i are half planes is for the case $k = 1$ contained by a limiting process in Theorem IX, and for all values of k has been established by Mr. B. Z. Linfield, in a paper to be published in these Transactions.

the supplementary angle all the circles C_i' which bound loci exterior to them, and which cuts at this same angle all the circles C_j'' which bound loci exterior to them and at the supplementary angle all circles C_j'' which bound loci interior to them. Then the locus of the roots of the jacobian of f_1 and f_2 is a number of circular regions bounded by circles $C_1, C_2, \cdots, C_n$ each of which is cut by every circle S at an angle equal or supplementary to the angles cut by S on C_i' and C_j''; the regions which are the loci of the roots of the jacobian may be either internal or external to their bounding circles. The circles $C_1, C_2, \cdots, C_n$ are included among the circles traced by the roots of the jacobian when all the roots of f_1 and f_2 are concentrated on the circles bounding their proper loci and move so that one of the circles S constantly passes through them all, while the lines tangent to these bounding circles C_i', C_j'' at the points which are the roots of f_1 and f_2 (and the lines tangent to C_k at the points which are the roots of the jacobian) all become parallel when S is transformed into a straight line. Any region C_i having no point in common with any other region C_j contains a number of roots of the jacobian equal to the multiplicity of the root of the jacobian which traces that circle C_i under these conditions.

We shall first undertake to prove this theorem for the simplest case, namely, that the circles S form a coaxial family having no point common to all those circles. We transform so that the circles S have a common center P. If the given circles C_i', C_j'' are all straight lines, all the regions C_i', C_j'' have a common point, the locus of the roots of the jacobian is the entire plane, and the theorem is proved. In any other case, all the circles $C_1', \cdots, C_{n'}', C_1'', \cdots, C_{n''}''$ are equal; the loci corresponding to the former can be considered to lie inside the bounding circles and those corresponding to the latter to lie outside the bounding circles.

We shall use the same method of proof as was used in § 9, namely, the replacing of all the particles in the field of force corresponding to the roots of f_1 by a single particle η_1, the replacing of all the particles corresponding to the roots of f_2 by a single particle η_2, and the study of the loci of η_1 and η_2. We shall prove that no matter what may be the location of the circular regions which are the loci or the distribution of the roots of the ground forms among these loci, *the locus of the roots of the jacobian is always the entire plane.*

If a point z is exterior to all the circles C_i', C_j'' and exterior to all the circles S which actually cut those circles C_i', C_j'', then z is a root of the jacobian. For if $n'' > 1$, z lies in C_1'' and C_2'', may lie at a multiple root of f_2, and hence is a point of the locus. If $n'' = 1$, replace the particles ζ_1 and ζ_2 of masses m_1' and m_2' whose loci are C_1' and C_2' by a single equivalent particle ζ of mass $m_1' + m_2'$ whose locus is a circular region S_1'. Then the circle S_1' is larger than C_1'; this follows from the fact that there are two circles through z tangent to S_1', C_1', C_2', and having the same kind of contact with all three of these circles; neither intersection of those two tangent circles with each other separates

any two of the points of tangency of the circles S_1', C_1', C_2'. The locus of ζ is the *interior* and boundary of S_1'. If $n' > 2$, we now replace ζ and the particle ζ_3 whose mass is m_3' and whose locus is C_3' by an equivalent particle ζ' of mass $m_1' + m_2' + m_3'$. The locus S_2' of ζ' is the interior of a circle which does not contain z and which is larger than C_3', by the reasoning just used. We continue in this way and finally replace all the particles representing the roots of f_1 by a single equivalent particle η_1 whose locus S_1 is larger than the circles C_i' (or if $n'' = 1$, equal to them). In any case, the region S_1 has at least one point in common with the region C_1'', so z is a point of the locus.

Denote by Σ_1 the larger and by Σ_2 the smaller of the two circles of the family S which are tangent to C_i' and C_j''. If z is interior to Σ_2, and if Σ_2 is exterior to C_i', C_j'', the reasoning just given applies with practically no change. If $n'' = 1$, the locus of η_1 which represents all the roots of f_1 is a region S_1 whose bounding circle is larger than the circles C_i', so the region S_1 must have at least one point in common with the region C_1'', and z is a point of the locus.

Let us now consider a point z between Σ_1 and Σ_2 under the assumption that Σ_2 is not interior to the circles C_i', C_j''. If $n'' = 1$, we find as before circles S_1', S_2', $\cdots$, S_1 all larger than C_1'. In fact, we need consider only points z interior to or on at most one circle C_i'. Describe a circle Σ through z and through the points of tangency of Σ_1 with C_1' and C_2'. When ζ_1 and ζ_2 lie at these two points, the point ζ corresponding lies on Σ, and is such that z and ζ separate ζ_1 and ζ_2. Hence ζ is exterior to Σ_1. Similarly there is a point ζ interior to Σ_2, so S_1' is indeed larger than C_1'. Thus we find z to be a point of the locus, for S_1 and C_1'' have a common point.

If $n'' > 1$, we need consider only points z interior to at most one circle C_i' and exterior to at most one circle C_j''. The circles S_1', S_2', $\cdots$, S_1 are all larger than C_1' (or equal to C_1' if $n' = 1$). The region S_1'' which is the locus of ξ, the particle equivalent to the particles ξ_1 and ξ_2 whose loci are C_1'' and C_2'', is a circular region which contains all the region common to C_1'' and C_2''. Hence the circle S_1'' is smaller than C_1''. The region S_1 which is the locus of the particle η_1 representing all the roots of f_1 is larger than each circular region C_i'. The region S_2 which is the locus of the particle η_2 representing all the roots of f_2 is bounded by a circle smaller than the bounding circle of each region C_i'', so S_1 and S_2 have at least one common point and z is a point of the locus.

It remains to consider points z interior to Σ_2, if Σ_2 is interior to C_i' and C_j'', but this treatment is so similar to the results already given that it is omitted. It remains also to consider points z on Σ_1 and on Σ_2, but since all other points of the plane are points of the locus and the locus of the roots of the jacobian is a closed point set, these points also belong to the locus. Theorem XI is now completely proved if the circles S have no point in common.

14. **Theorem XI, proof continued.** We next undertake to prove Theorem

XI for the case that the circles S form a coaxial family of circles all tangent at a single point P, which point we transform to infinity. If none of the original circular regions $C_1', \cdots, C_{n'}', C_1'', \cdots, C_{n''}''$ is the point at infinity, our results just proved for the case of circles S having no point in common hold without essential change; the entire plane is the locus of the roots of the jacobian. But P may be considered a null circle, the locus of a number of roots of f_1 and of f_2; in this case the entire plane need not be the locus of the roots of the jacobian.

If all the roots of f_2 are concentrated at P, then either all the roots of f_1 are also concentrated at P and the locus of the roots of the jacobian is the whole plane, or there are a number of fixed equal circles $C_1', \cdots, C_{n'}'$ bounding loci interior to them. In this latter case the field of force is precisely the field corresponding to Theorem X, so for this case Theorem XI is already proved. The case that there is at least one finite circle C_j'' requires some further consideration.

Denote by Σ_1 and Σ_2 the lines which belong to the coaxial family S and which are tangent to all the circles $C_1', \cdots, C_{n''}''$ and transform so that Σ_1 and Σ_2 are horizontal, with Σ_1 above Σ_2. Any point z on the boundary of the locus of the roots of the jacobian must lie on one of the circles traced by the roots of the jacobian when the roots of the ground forms trace the boundaries of their respective loci all constantly lying on one variable circle S and tracing the circles $C_1', \cdots, C_{n''}''$ in the same sense; this is the location of the roots of the ground forms described in the statement of Theorem XI. This fact is proved precisely as in §§ 9, 10, if z lies on a circle S which actually cuts the circles $C_1', \cdots, C_{n''}''$. We replace the particles at the roots of f_1 by a single equivalent particle η_1, the particles at the roots of f_2 by a single equivalent particle η_2, and notice that when z is on the boundary of its locus the loci of η_1 and η_2 cannot overlap. To complete the proof of Theorem XI in our special case it is sufficient to consider points z say above Σ_1 and to prove that all such points are points of the locus of the roots of the jacobian.

If z is a point above Σ_1 and if there are two or more finite circles C_j'', z is common to two or more of those circular regions and is therefore a point of the locus. If there is but one circle C_j'' other than at infinity, further consideration is required.

The locus of η_2 is the exterior of a circle S_2 obtained from C_1'' by similarity transformation with z as center, and S_2 is farther from z than is C_1''. Thus if there is but one finite circle C_1', the locus of η_1 is the interior of a circle S_1' obtained from C_1' by similarity transformation with z as center, and the loci S_1' and S_2 must have at least one common point, so z belongs to the locus.

If there are two finite circles C_i', we replace the particles whose loci are C_1' and C_2' by a single equivalent particle whose locus is S_1', and then replace that

particle and the particles at infinity which represent the roots of f_1 by a single equivalent particle whose locus is S_1. There are two circles through z tangent to S_1', C_1', C_2'; one of these circles contains S_1', C_1', C_2' and the other contains none of these circles. The external tangents to S_1' and C_1' intersect on the horizontal line through z, the radical axis of the circles through z tangent to S_1', C_1', C_2'.* Hence there is a circle Σ tangent to Σ_1 and Σ_2 and such that Σ and S_1' have z as common external center of similitude. It follows that the regions S_1 and S_2 have at least one common point. In fact, if there is no line through z which cuts both C_1'' and S_1', S_1 is entirely interior to S_2. If there is a line through z which cuts both C_1'' and S_1', a line through z and tangent to C_1'' cuts S_1' and S_1 and lies wholly in S_2. Thus z is a point of the locus.

If there are three finite circles C_i', we find S_1' as before; the external center of similitude of S_1' and C_3' lies on the horizontal line through z, so that line is the radical axis of the two circles through z and tangent to S_1', S_2', C_3'; one of these tangent circles contains S_1', S_2', C_3', and the other contains none of these three circles. Then the external center of similitude of S_2' and C_3' lies on the horizontal line through z, and as before we find that S_1 and S_2 have at least one common point. This reasoning is general for any number of circles C_i'.

Every point z above Σ_1 and hence every point below Σ_2 is a point of the locus. We may show either by similar reasoning or as in § 13 that every point on either of these lines belongs to the locus.

Theorem XI is thus proved for circles S all tangent at a single point. It is worth while, perhaps, to point out explicitly that there actually exist situations with one or more circles C_j'', and where the locus of the roots of the jacobian is not the entire plane. Thus, let there be merely two finite circles C_j'' and let z be interior to both of them. Then S_1'' is a region exterior to a circle which surrounds z. If z is exterior to all the circles C_i' and if the locus S_1 is interior to its bounding circle, it is possible so to choose the number of roots of f_2 at infinity that S_2 shall be the region exterior to a circle which entirely contains S_1. Then S_1 and S_2 have no common point and z is not a point of the locus of the roots of the jacobian.

15. **Theorem XI; completion of the proof.** The case that the circles S of Theorem XI form a coaxial family of circles through two distinct points P and P' remains to be dealt with. Transform P' to infinity. The points P and P' are considered as null circles and hence allowed to be loci of a number of roots of f_1 or f_2 or both. As in § 8 we may assume that there is at least one circle C_i' or C_j'' distinct from P and P'.

If the circles $C_1', \cdots, C_{n''}''$ surround P, Theorem XI can be proved precisely as in §§ 8–10. If $C_1', \cdots, C_{n''}''$ do not surround P, these same methods show that no point z is on the boundary of its locus unless z is on one of the circles

* Coolidge, *A Treatise on the Circle and the Sphere*, p. 111, Theorem 217.

described in the theorem, provided that there is a circle S through z which actually cuts all the circles $C_1', \cdots, C_{n''}''$. If all the finite regions $C_1', \cdots, C_{n''}''$ are interior to their bounding circles, the theorem is Theorem VI and completely proved. If two or more of these regions are exterior to their bounding circles, every point z not on a circle S which cuts all the circles $C_1', \cdots, C_{n''}''$ is a possible position of pseudo-equilibrium and hence a point of the locus. It remains to consider the case of such points z with merely one finite region, say C_1', exterior to its bounding circle. Let the line of centers of the circles $C_1', \cdots, C_{n''}''$ be horizontal and denote by Σ_1 and Σ_2 the common tangents to $C_1', \cdots, C_{n''}''$. Let C_1' lie to the left of P. We shall phrase the proof for $n' > 1, n'' > 1$.

The particles ζ_1 and ζ_2 whose loci are C_1' and C_2' are to be replaced by a particle ζ whose locus is a circular region S_1'. There are two circles through z tangent to C_1', C_2', S_1', one of which includes C_1' but not C_2', the other of which includes C_2' but not C_1'. If the locus S_1' is not the entire plane, it follows from a simple consideration of points ζ_1, ζ_2, ζ on the circle through z orthogonal to C_1' and C_2' that these two tangent circles include S_1' and exclude S_1' respectively. If the locus S_1' is the entire plane, the loci $S_2', S_3', \cdots, S_1$ are all the entire plane and z is a point of the locus.

These two tangent circles intersect on the line Pz,* and the circle S_1' lies to the left of Pz. The external center of similitude of C_1' and S_1' and the internal center of similitude of C_2' and S_1' lie on Pz. It is thus true that the external center of similitude of S_1' and any circle C_j'' lies on Pz and that the internal center of similitude of S_1' and any circle C_i' other than C_1' lies on Pz.

We now replace ζ and ζ_3, the particle whose locus is C_3', by a single equivalent particle ζ' whose locus is a circular region S_2'. If the locus S_2' is not the entire plane, there are two circles through z which intersect on Pz and which are tangent to S_1', C_3', S_2'; one of these tangent circles contains S_1' and S_2' but does not contain C_3', the other contains C_3' but neither S_1' nor S_2'. We continue in this way and finally reach a circle S_1 which bounds the locus of the point η_1 which represents all the roots of f_1; the locus of η_1 is exterior to S_1. The external center of similitude of S_1 and any of the finite circles C_j'' (and also of C_1') lies on Pz, and the internal center of similitude of S_1 and any of the finite circles C_i' except C_1' lies on Pz.

Similarly the particles representing the roots of f_2 are replaced by a single equivalent particle η_2 whose locus is the interior of a circle S_2 such that the external center of similitude of S_2 and any of the finite circles C_j' lies on Pz.

Hence the external center of similitude of S_1 and S_2 lies on Pz, from which it follows as before that there is at least one point common to S_1 and S_2, so z is a point of the locus. Likewise all points of Σ_1 and Σ_2 are points of the locus.

* Coolidge, loc. cit.

As in § 14, cases actually arise here where all the regions C_1', $\cdots$, $C_{n''}''$ are not within their finite bounding circles and yet the locus of z is not the entire plane; the proof is as in § 14.

The number of roots of the jacobian in a region C_i which has no point in common with any other region C_i which is a part of the locus of the roots of the jacobian can be determined as in § 10 for Theorem VI; the proof of Theorem XI is now complete.

The determination in Theorem XI of whether or not a given circle C_i is actually a part of the boundary of the locus of the roots of the jacobian, and if so whether the circular region corresponding lies interior or exterior to C_i, can be made in any given case by the methods developed in the present chapter.

Theorem XI has obvious applications which will easily be made by the reader to the study of the location of the roots of the derivative of a polynomial and of the derivative of a rational function.

Chapter III: On centers of gravity

16. **The loci of certain centers of gravity.** There is a striking analogy between some of our results concerning the location of the roots of the jacobian of two binary forms and results which are easily proved concerning the location of the center of gravity of a number of particles. Thus, the fact that if a number of positive particles lie in a circle their center of gravity also lies in that circle is analogous to Lemma I (II, p. 102) and was used in the proof of that lemma, and is also analogous to the theorem of Lucas. From this fact and Theorem VIII of III we prove the analogue of a theorem given in II, p. 115 (= Theorem I of S) precisely as that theorem was proved:

Theorem XII. *If the interiors and boundaries of two circles C_1 and C_2 of centers α_1 and α_2 and radii r_1 and r_2 are the loci respectively of m_1 and m_2 unit positive particles, then the locus of the center of gravity of these particles is the interior and boundary of the circle C whose center is*

$$\frac{m_1 \alpha_1 + m_2 \alpha_2}{m_1 + m_2}$$

and whose radius is

$$\frac{m_1 r_1 + m_2 r_2}{m_1 + m_2}.$$

The three circles C_1, C_2, C have as common external center of similitude the point

$$\frac{r_1 \alpha_2 - r_2 \alpha_1}{r_1 - r_2}.$$

Theorem XII can be largely extended by the method of proof used for III, Theorem VIII in S:

THEOREM XIII. *If the interiors and boundaries of n circles C_i, whose centers are α_i and radii r_i, are the loci respectively of m_i unit positive particles, then the locus of the center of gravity of these particles is the interior and boundary of the circle C whose center is*

$$\alpha = \frac{m_1 \alpha_1 + m_2 \alpha_2 + \cdots + m_n \alpha_n}{m_1 + m_2 + \cdots + m_n}$$

and whose radius is

$$r = \frac{m_1 r_1 + m_2 r_2 + \cdots + m_n r_n}{m_1 + m_2 + \cdots + m_n}.$$

Denote the m_i particles in or on C_i by $z_1^{(i)}, z_2^{(i)}, \cdots, z_{m_i}^{(i)}$, so that $|z_j^{(i)} - \alpha_i| \leqq r_i$ for every i and j. The center of gravity of all the $m_1 + m_2 + \cdots + m_n$ particles is

$$z = \frac{(z_1^{(1)} + z_2^{(1)} + \cdots + z_{m_1}^{(1)}) + \cdots + (z_1^{(n)} + z_2^{(n)} + \cdots + z_{m_n}^{(n)})}{m_1 + m_2 + \cdots + m_n},$$

so that we have

$$z - \alpha = \frac{\begin{array}{l}[(z_1^{(1)} - \alpha_1) + (z_2^{(1)} - \alpha_1) + \cdots + (z_{m_1}^{(1)} - \alpha_1)] + \cdots \\ \qquad + [(z_1^{(n)} - \alpha_n) + (z_2^{(n)} - \alpha_n) + \cdots + (z_{m_n}^{(n)} - \alpha_n)]\end{array}}{m_1 + m_2 + \cdots + m_n}$$

and hence z is on or within C.

Conversely, if z is given on or within C, we determine $z_j^{(i)}$ by the relation

$$z_j^{(i)} - \alpha_i = (z - \alpha)\frac{r_i}{r},$$

and we have the $z_j^{(i)}$ satisfying the proper conditions. The proof is thus complete. It may be remarked that when the $z_j^{(i)}$ trace their proper circles in such a manner that $(z_j^{(i)} - \alpha_i)/r_i$ is the same for every i and j, then this common value is equal to $(z - \alpha)/r$ while z traces its circle C.

Theorem XIII can be extended without difficulty in various directions: to particles of negative or even complex mass; to space of any number of dimensions; to give a result which shall be invariant under linear transformation; to regions other than the interiors of circles, especially convex regions. In this last extension, use is made of the fact that if m_i particles lie in a convex region their center of gravity also lies in that region; hence such results as III, Theorem IX can be applied.

There is much more than a mere analogy between Theorems XII and XIII for centers of gravity and our previous results concerning the derivatives of polynomials. In fact, the only root of the $(n - 1)$st derivative of a polynomial of degree n lies at the center of gravity of the roots of that polynomial. When viewed in this light, Theorems XII and XIII are results relating to the

location of the roots of the derivatives of a polynomial even if not of the jacobian of two binary forms, and are conceived in precisely the same spirit as is Theorem IX. Thus the entire discussion of § 5 holds practically without change if we consider the problem of determining the locus of the roots of the kth derivative of a polynomial of degree n whose roots have certain assigned circular regions as their loci. Theorem XIII gives the complete solution of that problem for $k = n - 1$ if the assigned circular regions are interior to their bounding circles.

As a particular case of Theorem XIII, the fact that if a number of particles lie in a convex region their center of gravity also lies in that region follows from the theorem of Lucas* as applied successively to the various derivatives of a polynomial.

17. **The center of gravity of the roots of the derivative of a rational function and of the jacobian of two binary forms.** The center of gravity of any set of points has interesting properties with reference to that point set. It furnishes, for example, an approximate idea of the location of those points. Any line through the center of gravity either passes through all the points of the set or separates at least two of them.† We shall now find some results connecting the centers of gravity of related polynomials of the sort we have been considering. A classical theorem of this nature follows from a remark previously made:

THEOREM XIV. *The center of gravity of the roots of a polynomial coincides with the center of gravity of the roots of the derived polynomial.*

We derive the corresponding result for a rational function, which we take in the form

$$f(x) = \frac{x^m + a_0 x^{m-1} + a_1 x^{m-2} + \cdots + a_{m-1}}{x^n + b_0 x^{n-1} + b_1 x^{n-2} + \cdots + b_{n-1}};$$

$$f'(x) = \frac{\begin{array}{l}(x^n + b_0 x^{n-1} + \cdots)(mx^{m-1} + (m-1)a_0 x^{m-2} + \cdots) \\ \qquad - (x^m + a_0 x^{m-1} + \cdots)(nx^{n-1} + (n-1)b_0 x^{n-2} + \cdots)\end{array}}{(x^n + b_0 x^{n-1} + \cdots)}.$$

If we denote by α the center of gravity of the finite roots of $f(x)$ and by β that of the finite poles of $f(x)$, if $m \neq n$ and if $f(x)$ has no finite multiple poles, we have for the center of gravity of the finite roots of $f'(x)$ the formula

$$\gamma = -\frac{(m-n-1)a_0 + (m-n+1)b_0}{(m+n-1)(m-n)}$$

$$= \frac{m(m-n-1)\alpha + n(m-n+1)\beta}{(m+n-1)(m-n)},$$

* On the other hand, Lemma I (II, p. 102) enables us to use this fact to give immediately a very simple proof of the theorem of Lucas.

† An application of this fact to the more precise location of the roots of algebraic equations is given by Laguerre, *Œuvres*, vol. I, pp. 56, 133.

which is a point collinear with α and β. If α and β coincide, γ coincides with them; if $m = n + 1$, γ coincides with β; if $m = n - 1$, γ coincides with α. If $n = 0$, we have Theorem XIV. If $m = n$, γ cannot be expressed in terms of merely α and β, as is found simply by computing γ.

Let us inquire in what respect this work on centers of gravity can be made invariant under linear transformation and can be applied to the jacobian of two binary forms.

The concept *center of gravity* is surely not invariant under linear transformation. In fact, given any two distinct points of the plane ξ and η, any third point ζ of the plane can, by a suitable transformation, be made to correspond to the center of gravity of the transformed ξ and η. We need simply to transform to infinity the harmonic conjugate of ζ with respect to ξ and η.

We cannot expect to obtain results with the ordinary definition of center of gravity, so we introduce a new definition. The point G is said to be the *centroid of a set of points with respect to P* if when P is transformed to infinity G transformed into the center of gravity of the points corresponding to the original set. We suppose that P is not a point of that set. It should be noted by way of justification of the definition that the point G is uniquely defined, since the center of gravity is invariant under similarity transformation. The relation between the points P and G is not reciprocal.

The centroid with respect to a point of a set of points gives a rough indication of the distribution of that set of points, like the ordinary center of gravity. In particular, if P is external to a circular region containing the set of points, G is also in that circular region. In fact, examination of the proof of Lemma I (II, p. 102) will show that the force at a point P external to a circular region C due to k particles in C is equivalent to the force at P due to k particles which coincide at a point Q in C, and Q is the centroid of the k particles with respect to P. Thus we are studying the relation between P, Q, and the k particles, which is the same as Laguerre's relation set up between those points, referred to in § 2.

Let f_1 and f_2 be two binary forms, of respective degrees p_1 and p_2, and let the point P at infinity be a k-fold root of f_1. Let α, β, γ be the centroids with respect to P of the roots of f_1 other than P, the roots of f_2 (all of which are supposed finite), the finite roots of the jacobian of f_1 and f_2, respectively. We easily find that

$$\gamma = \frac{(p_1 - k)(k + 1)\alpha - (p_1 - kp_2)\beta}{k(p_1 + p_2 - k - 1)},$$

a point collinear with α and β. If α and β coincide, γ coincides with them; if $p_1 = kp_2$, $\gamma = \alpha$; if $p_1 = k$, $\gamma = \beta$, which is Theorem XIV. Always we shall have*

* It might seem at first sight that this cross ratio should be p_1 / k, since by Lemma II

$$(P, \alpha, \beta, \gamma) = \frac{(p_1 - k)(k+1)}{k(p_1 + p_2 - k - 1)},$$

which expresses the entire result in invariant form.

Chapter IV: On the roots of the jacobian of two real forms

18. **The locus of the roots of the derivatives of a polynomial whose roots are real.** The present chapter is devoted mainly to general theorems of the kind developed in Chapter II, but where we restrict ourselves to ground forms whose coefficients are real or can be made real by suitable linear transformation. We are placing additional restrictions on our ground forms, so it is to be expected that some additional properties will appear.

Any result concerning the location of the roots of the derivative of a polynomial is also a result concerning the roots of the jacobian of two binary forms. Thus all the facts proved in A can be given this interpretation and other results can be found by linear transformation.* The reader can easily formulate these new theorems. We now prove a new result concerning the derivatives of polynomials all of whose roots are real.

Theorem XV. *Let intervals I_i $(i = 1, 2, \cdots, m)$ of the axis of reals, whose end points are α_i, β_i, $\alpha_i \leqq \beta_i$, be the respective loci of m_i roots of a polynomial $f(z)$ which has no other roots. Then the locus of the roots of $f^{(k)}(z)$ is composed of a number of intervals $I_i^{(k)}$ of the axis of the reals. The left-hand end points of the intervals $I_i^{(k)}$ are the roots of $f^{(k)}(z)$ when the roots of $f(z)$ are concentrated at the points α_i; the right-hand end points are the corresponding roots of $f^{(k)}(z)$ when the roots of $f(z)$ are concentrated at the points β_i. Any interval $I_i^{(k)}$ which has no point in common with any other interval $I_j^{(k)}$ contains a number of roots of $f^{(k)}(z)$ equal to the multiplicity of its left-hand end point as a root of $f^{(k)}(z)$ when the roots of $f(z)$ are the points α_i. If the intervals I_i are all of equal length, the intervals $I_i^{(k)}$ are of this same length. If there is a point P which is a center of similitude for every pair of the intervals I_i (which is always true if $m = 2$), P is also a center of similitude for every pair of intervals $I_i^{(k)}, I_j^{(k)}$.*†

We prove this theorem under the assumption that no interval I_i reduces to

(II, p. 102) when the $p_1 - k$ finite roots of f_1 coalesce at α and the p_2 finite roots of f_2 coalesce at β there is but one position of equilibrium, namely, at the point γ' such that $(P, \alpha, \beta, \gamma') = p_1/k$. However, the jacobian vanishes not only at γ' but also at α and β if $p_1 - k$ and p_2 are greater than unity. It is the centroid with respect to P of *all* the finite roots of the jacobian that we have denoted by γ. The two formulas are the same when $p_1 - k = 1$, $p_2 = 1$.

*(Added in proof): There is an error in the statement of the italicized theorem of A, p. 133, as has been pointed out by Nagy, Jahresbericht der Deutschen Mathematiker-Vereinigung. vol. 31 (1922), pp. 245, 246. That theorem has no meaning as it appears at present, but becomes correct if the word *exterior* is replaced by the word *other*. The theorem is correctly stated in the abstract of A, Bulletin of the American Mathematical Society, vol. 26 (1919–20), p. 259.

† Some special cases of this theorem are given by Nagy, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 27 (1918), pp. 37–43; 44–48. The special case $m = 2$, $k = 1$ is Theorem II of S.

a point; to include this more general case requires merely a slight change in phraseology. We prove the theorem first for the case $k = 1$. In the theorem the intervals are assumed to be finite, but the theorem can be extended to include infinite intervals.

Let us denote by $\alpha_i^{(k)}$ the roots of $f^{(k)}(z)$ when the roots of $f(z)$ are concentrated at the points α_i, $\alpha_i^{(k)} \leqq \alpha_j^{(k)}$ when $i < j$, and similarly by $\beta_i^{(k)}$ the roots of $f^{(k)}(z)$ when the roots of $f(z)$ are concentrated at the points β_i, $\beta_i^{(k)} \leqq \beta_j^{(k)}$ when $i < j$. The intervals $I_i^{(k)} : (\alpha_i^{(k)}, \beta_i^{(k)})$ are then to be proved to form the locus of the roots of $f^{(k)}(z)$.

Let us start with the roots of $f(z)$ concentrated at the points α_i, and move these roots continuously to the right until they reach the points β_i. The roots of $f'(z)$ also vary continuously in their totality; they start at the points α_i' and reach the points β_i'. If we number these roots, commencing at the left, we can even say that the nth root z_n' of $f'(z)$ varies continuously. We now prove that z_n' moves always to the right.

The equation determining z_n' is of the form

$$F = \frac{m_1}{z_n' - \gamma_1} + \frac{m_2}{z_n' - \gamma_2} + \cdots + \frac{m_m}{z_n' - \gamma_m} = 0, \tag{3}$$

where the γ_i are the roots of $f(z)$, coinciding in any multiplicities m_i desired. We compute the values

$$\frac{\partial F}{\partial z_n'} = -\frac{m_1}{(z_n' - \gamma_1)^2} - \frac{m_2}{(z_n' - \gamma_2)^2} - \cdots - \frac{m_m}{(z_n' - \gamma_m)^2},$$
$$\frac{\partial F}{\partial \gamma_i} = \frac{m_i}{(z_n' - \gamma_i)^2}.$$

It is always true that $\partial z_n'/\partial \gamma_i$ is positive, so z_n' always increases with γ_i.

Equation (3) is no longer valid to determine z_n' if z_n' is located at a multiple root of $f(z)$. Under these circumstances, if γ_i does not coincide with z_n', the motion of γ_i does not change the position of z_n'. If γ_i does coincide with z_n' and if γ_i is moved to the right, z_n' is either unchanged or moved to the right; this follows immediately from the fact that a k-fold root of $f(z)$ is a $(k-1)$-fold root of $f'(z)$ and from the fact that every interval bounded by roots of $f(z)$ contains at least one root of $f'(z)$.

From the general fact, then, that the nth root z_n' of $f'(z)$ varies continuously and in one sense under the indicated variation in the roots of $f(z)$, it follows that z_n' traces the entire interval from α_n' to β_n'. It remains to be shown that z_n' can never be outside of the interval (α_n', β_n'). If we assume z_n' to lie outside of that interval, say for definiteness to the right, for some possible position of the roots of $f(z)$, motion of those roots of $f(z)$ to the right always within their proper loci would move z_n' to the right and when the roots of $f(z)$ reached

the ends of their proper intervals z_n' would lie to the right of β_n', which is impossible.

The determination of the locus in Theorem XV is now complete for $k = 1$; the statement relative to the number of roots of $f'(z)$ in the various intervals is readily proved by the continuity methods previously used.

For the case of $k = 2$, the continuity of the motion of the roots of $f''(z)$ due to the motion of the roots of $f(z)$ shows that every point of each of the intervals I_k'' is a root of $f''(z)$ for some $f(z)$. No other point can be a root of $f''(z)$, for when the roots of $f(z)$ vary continuously in one sense, the roots of $f'(z)$ and therefore of $f''(z)$ vary continuously in that same sense. The number of roots of $f''(z)$ proper to the various intervals is as indicated. Continuance of the method of reasoning enables us to determine the locus for $k = 3$ and so on for the other values of k.

If all the intervals I_i are of the same length, the $f(z)$ whose roots are the β_i is obtained from the $f(z)$ whose roots are the α_i by a translation, so the β_i are obtained from the corresponding α_i by the same translation and the I_i' (and hence the $I_i^{(k)}$) are all of the same length as the I_i. If the β_i are obtained from the α_i by a similarity transformation, the $\beta_i^{(k)}$ are obtained from the $\alpha_i^{(k)}$ by the same transformation.

19. **The extension of theorems for the derivative of a polynomial to the roots of the jacobian.** Theorem XV cannot be immediately extended to the location of the roots of the jacobian of two binary forms, where the loci of the roots of both forms are intervals of the axis of reals. First, all the roots of both forms may coincide, so that the locus of the roots of the jacobian is not a number of intervals of the axis of reals. Second, the jacobian may have non-real roots even when it does not vanish identically.*

We can avoid this first possibility by requiring that the loci of the roots of f_1 and f_2 be so arranged that the two forms cannot be identically equal. We can avoid the second possibility by requiring that these loci be so arranged that no two roots of f_1 can separate two roots of f_2. Then all the roots of the jacobian are real, for on any interval bounded by roots of either form and containing no root of the other form there lies at least one root of the jacobian.

With these new restrictions, Theorem XV extends directly to the jacobian of two binary forms. If all the intervals which are the loci of the roots of both forms are finite, we consider the α_i (β_i) to be at the left-hand (right-hand) ends of those intervals which are loci of the roots of f_1 and at the right-hand (left-hand) ends of those intervals which are loci of the roots of f_2. For infinite intervals this notation is reversed. The locus of the roots of the jacobian is composed of the intervals whose end points are the corresponding

* This is shown by the simplest examples, such as $f_1 = z_1^2 - z_2^2$, $f_2 = z_1 z_2$, $J = 2(z_1^2 + z_2^2)$.

roots of the jacobian when the roots of the ground forms are respectively the α_i and the β_i.

A special case of this result is so similar to Theorem II that it deserves to be stated explicitly:

THEOREM XVI. *Let f_1 and f_2 be binary forms of degrees p_1 and p_2 respectively, and let arcs A_1, A_2, A_3 of a circle C be the respective loci of m roots of f_1, the remaining $p_1 - m$ roots of f_1, and all the roots of f_2. Suppose A_3 to have not more than one point in common with A_1 nor with A_2 and no point in common with both A_1 and A_2. Denote by A_4 the arc of C which is the locus of points z_4 such that*

$$(z_1, z_2, z_3, z_4) = \frac{p_1}{m},$$

when z_1, z_2, z_3 have the respective loci A_1, A_2, A_3. Then the locus of the roots of the jacobian of f_1 and f_2 is composed of the arc A_4 together with the arcs A_1, A_2, A_3, except that among the latter the corresponding arc is to be omitted if any of the numbers m, $p_1 - m$, p_2 is unity. If an arc A_i $(i = 1, 2, 3, 4)$ has no point in common with any other of those arcs which is a part of the locus of the jacobian, it contains precisely $m - 1$, $p_1 - m - 1$, $p_2 - 1$, or 1 of those roots according as $i = 1, 2, 3$, or 4.

Theorem XVI, as a special case of our more general result on the location of the roots of the jacobian, needs no separate proof, but it is interesting to notice that it can be proved in precisely the same manner as Theorem II was proved. Theorem I in the proof of Theorem II is replaced by III, Theorem IV, and Lemma I (II, p. 102) is replaced by the following

LEMMA. *The force at a point P on a circle C due to k unit positive particles lying on an arc A of C not containing P is equivalent to the force at P due to k coincident particles lying on A.*

We shall now obtain a result which has some relation to Theorem XVI as well as to Jensen's theorem, proved in A. We are dealing with pairs of points inverse with respect to a line, and as in A shall term circles whose diameters are the segments joining such pairs of points *Jensen circles*. Let f_1 and f_2 be two real forms which have not necessarily all their roots real. Let finite or infinite segments I_1, I_2, I_3 of the axis of reals either contain respectively m roots of f_1, the remaining $p_1 - m$ roots of f_1, and the p_2 roots of f_2, or contain some of these roots and the intercepts on the axis of reals of the Jensen circles of the remainder. Then any real root of the jacobian of f_1 and f_2 which is exterior to I_1, I_2, and I_3 lies in the interval I_4 which is the locus of the point z_4 defined by

$$(z_1, z_2, z_3, z_4) = \frac{p_1}{m}$$

when I_1, I_2, I_3 are the respective loci of z_1, z_2, z_3.

To prove this result we require the preceding lemma and the fact that the force at a point due to two particles is equivalent to the force at that point due to two coincident particles situated at the harmonic conjugate of that point with respect to the other two. If a point P is exterior to I_i, its harmonic conjugate with respect to two points the intersections of whose Jensen circle with the axis of reals lie in I_i, also lies in I_i.

This result is not expressed so as to be invariant under linear transformation, for if a real linear transformation is made and if the point at infinity is not invariant the Jensen circles are not invariant.

Our final result on the roots of the jacobian is similarly not invariant under linear transformation; it can be proved from the fact proved in A, § 2, that the force due to two positive particles at a point above the axis of reals but interior to their Jensen circle has a component vertically downward; at a point above the axis of reals but exterior to the Jensen circle the force has a component vertically upward.

THEOREM XVII. *If the forms f_1 and f_2 are both real and if f_1 has no finite real root, there is no root of the jacobian of f_1 and f_2 exterior to all the Jensen circles corresponding to the roots of f_2 but interior to all the Jensen circles corresponding to the roots of f_1.*

20. **Conclusion: extension of results to other types of polynomials.** We have considered in this paper generalizations of Theorem II in various directions. There is still another direction which we have not mentioned, namely, to the roots of polynomials other than the jacobian of two binary forms or the derivatives of a polynomial.

Thus the jacobian of two forms f_1 and f_2, of respective degrees p_1 and p_2, all of whose roots are finite and which correspond to two polynomials ϕ_1 and ϕ_2, has the same roots as the polynomial

$$p_2 \phi_1' \phi_2 - p_1 \phi_1 \phi_2'.$$

If we set ϕ_1 equal to the product of two polynomials ψ_1 and ψ_2 of respective degrees m and $p_1 - m$, Theorem II refers to the roots of the polynomial

$$p_2 \psi_1' \psi_2 \phi_2 + p_2 \psi_1 \psi_2' \phi_2 - p_1 \psi_1 \psi_2 \phi_2', \tag{4}$$

when the roots of ψ_1, ψ_2, ϕ_2 have the respective loci C_1, C_2, C_3.

We shall generalize Theorem II by considering three polynomials ω_1, ω_2, ω_3, of respective degrees μ_1, μ_2, μ_3, whose roots have the respective loci C_1, C_2, C_3. Our conclusion concerns the polynomial

$$\lambda_1 \omega_1' \omega_2 \omega_3 + \lambda_2 \omega_1 \omega_2' \omega_3 + \lambda_3 \omega_1 \omega_2 \omega_3', \tag{5}$$

where λ_1, λ_2, λ_3 are real* numbers not all zero such that

$$\lambda_1 \mu_1 + \lambda_2 \mu_2 + \lambda_3 \mu_3 = 0.$$

* Proof of Theorem I for complex λ enables us to remove this restriction of reality. See Walsh, Rendiconti del Circolo Matematico di Palermo, vol. 46 (1922), pp. 236–248.

It will be noticed that the polynomial (5) is indeed a generalization of (4), and has the additional advantage of being symmetric in $\omega_1, \omega_2, \omega_3$.

If a point z is a root of (5) yet exterior to C_1, C_2, C_3, we must have

$$\lambda_1 \frac{\omega_1'}{\omega_1} + \lambda_2 \frac{\omega_2'}{\omega_2} + \lambda_3 \frac{\omega_3'}{\omega_3} = 0,$$

$$\frac{\lambda_1 \mu_1}{z - \alpha_1} + \frac{\lambda_2 \mu_2}{z - \alpha_2} + \frac{\lambda_3 \mu_3}{z - \alpha_3} = 0,$$

by Lemma I (II, p. 102), where $\alpha_1, \alpha_2, \alpha_3$ lie in C_1, C_2, C_3 respectively. Hence z is given by the cross ratio

$$(\alpha_1, \alpha_2, \alpha_3, z) = -\frac{\lambda_3 \mu_3}{\lambda_1 \mu_1},$$

and lies in the region C_4 of Theorem I corresponding to the value

$$\lambda = -\frac{\lambda_3 \mu_3}{\lambda_1 \mu_1}.$$

We leave it to the reader to verify that the locus of the roots of (5) is composed of C_4 together with the regions C_1, C_2, C_3, except that among the latter the corresponding region is to be omitted if any of the degrees μ_1, μ_2, μ_3 is unity. If a region C_i $(i = 1, 2, 3, 4)$ has no point in common with any other of those regions which is a part of the locus of the roots of the jacobian, it contains precisely $\mu_1 - 1, \mu_2 - 1, \mu_3 - 1, 1$ of those roots according as $i = 1, 2, 3, 4$.

Many of the other theorems of the present paper, such as Theorems VI–XI, can similarly be extended to polynomials other than the jacobian of two binary forms or the derivative of a polynomial. Modifications of the methods used here can be made to apply to a still much broader type of polynomial about which the writer hopes to give some further results.

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I-CONJUGATE OPERATORS OF AN ABELIAN GROUP*

BY

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I. Introduction

Two operators of any group G are said to be I-conjugate if they correspond in at least one of the possible automorphisms of G. Every characteristic subgroup of G includes all the I-conjugates of each of its operators, and if a subgroup includes all the I-conjugates of its operators it is characteristic. In the present article it will be assumed that G is abelian. As two operators of any abelian group are I-conjugate if their prime power constituents have this property, and vice versa, it will only be necessary to consider the case when the order of G is of the form p^m, p being a prime number. Hence this will be done in what follows unless the contrary is stated.

Two fundamental questions in regard to the I-conjugate operators of G are: how many sets of I-conjugate operators are there in any abelian group, and how many operators are found in each of these sets? Both of these questions are answered in what follows, and the method for determining these numbers which is developed here seems to be as direct as possible. It is evident that every two I-conjugate operators are of the same order, and that a necessary and sufficient condition that every two operators of G which are of the same order be also I-conjugate is that all the invariants of G be equal to each other.

Two definitions of independent generators of G are in common use. According to one of these definitions the operators $s_1, s_2, \cdots, s_\lambda$ are called a set of independent generators of G whenever they satisfy the two conditions that they generate G and that no $\lambda - 1$ of them generate G. The number λ is known to be an invariant of G. According to the second definition, these λ operators must satisfy the additional condition that the subgroup generated by an arbitrary subset of them have only identity in common with the subgroup generated by the rest of these operators. According to the first definition, all the operators of G which do not appear in any one of the possible sets of independent generators of G constitute a subgroup of G known as its ϕ-subgroup, while these operators constitute such a subgroup according to the second definition when and only when the ratio of the largest invariant to the smallest invariant of G does not exceed p.

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To distinguish between sets of independent generators satisfying the first, or also the second, of these definitions, the latter are called *reduced sets of independent generators*. The former sets of independent generators are usually the most convenient when only questions relating to subgroups are considered, while the latter are more convenient in the study of conjugacy. In the present article it will be assumed that the sets of independent generators under consideration are reduced sets unless the contrary is stated. It will be seen that all the operators of G which do not appear in any such set generate a subgroup which includes all the independent generators of G except those of highest order and those whose order is equal to this highest order divided by p, if any of the latter exist.

When G has independent generators of different orders, its independent generators which are of the same order are evidently I-conjugate and can be selected from a set of I-conjugate operators of G which has no operator in common with the group generated by the remaining independent generators of the set. The latter independent generators can usually be selected in a large number of different ways and the subgroups which such operators generate may differ, but none of the subgroups can involve an operator of the set of I-conjugate operators from which the former independent generators must be chosen.

The number of the subgroups of G which are separately generated by all the independent generators of G which are of the same fixed order in its various possible sets of independent generators can easily be determined. In fact, it is the quotient obtained by dividing the number of ways in which the independent generators of such a subgroup can be selected from the operators of G by the number of ways in which these generators can be selected from the operators of one of these subgroups. If p^α is the order of such an independent generator the totality of the operators of order p^β, $0 \leqq \beta \leqq \alpha$, contained in all of these subgroups constitutes a single set of I-conjugate operators of G. Hence the distinct operators in all of these subgroups constitute the operators of α I-conjugate sets of G excluding identity. Two such subgroups corresponding to different values of α can have only identity in common, and G is the direct product of an arbitrary set of subgroups such that one and only one of the subgroups of this set corresponds to a particular possible value of α.

II. I-REDUCED OPERATORS OF A GROUP

It has been noted that when G contains a set of independent generators composed of λ_1 operators of order p^{α_1}, λ_2 operators of order p^{α_2}, $\cdots$, λ_γ operators of order p^{α_γ}, so that

$$\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \cdots + \lambda_\gamma \alpha_\gamma = m,$$

then the number of its sets of I-conjugate operators which are separately

powers of possible independent generators is $\alpha_1 + \alpha_2 + \cdots + \alpha_\gamma$, exclusive of identity. Each of these sets contains one and only one operator which satisfies both of the following conditions: It is the lowest possible power of an operator in the set of independent generators $s_1, s_2, \cdots, s_\lambda$ which appears in the former set, and this operator has the smallest possible subscripts. Such an operator will be called an *I-reduced operator* and hence each of the given sets of I-conjugate operators contains one and only one I-reduced operator, and this is a power of an independent generator of G.

In general, an I-reduced operator is defined as the single operator of a set of I-conjugate operators of G which satisfies the following conditions: It involves powers of the smallest possible number of constituents which are separately powers of the operators $s_1, s_2, \cdots, s_\lambda$ for the set of I-conjugate operators in which it is found, each of these constituents is raised to the lowest possible power, and the subscripts of operators of the set $s_1, s_2, \cdots s_\lambda$ of which these constituents are powers are as small as possible. A necessary and sufficient condition that an I-reduced operator involve powers of more than one of the operators $s_1, s_2, \cdots, s_\lambda$ is that all these powers be of different orders, and that the larger of two generators involved be raised to a higher power of p than the smaller, and this power have a larger order than the power of the smaller. In particular, no two of these constituents are powers of independent generators whose orders have a ratio which is less than p^2.

As each of the possible sets of I-conjugate operators of G is completely determined by the I-reduced operator which appears in the set, it results that the determination of the number of different sets of I-conjugate operators is equivalent to the determination of the possible number of different I-reduced operators. It should be noted that the number of I-reduced operators depends only upon the orders and the number of the different orders of the independent generators of G. That is, if G has more than one independent generator of the same order, the number of I-reduced operators of G is the same as that of the group having only one of these generators and only one generator whose order is equal to the order of every other independent generator of G.

To determine the number of operators of G which are I-conjugate with a given I-reduced operator T of G, it is convenient to call t-generators all the independent generators of G whose orders are equal to the orders of those independent generators whose powers appear in this I-reduced operator. The remaining independent generators of G will be called s-generators. Let $p^{\beta_1}, p^{\beta_2}, \cdots, p^{\beta_\theta}$ be the indices, in descending order of magnitude, of the various powers of t-generators which appear as constituents of T, and construct a subgroup of G whose independent generators are powers of s-generators which are determined as follows:

All the s-generators of G whose orders exceed the order of the largest

t-generator are raised to powers such that the common order of these powers is equal to that of the p^{β_1} power of this t-generator, and all the s-generators whose orders are smaller than the smallest t-generator are raised to the p^{β_θ} power. Each of the other s-generators of G is raised to the highest power whose index does not exceed the index of the power to which the next larger t-generator is raised to obtain a constituent of T and whose order is not less than the order of the power of the next lower t-generator which appears in T. The powers of the s-generators thus determined constitute a set of independent generators of the subgroup in question.

To obtain all the operators of the set of I-conjugate operators of T, we multiply all the operators of the subgroup noted in the preceding paragraph by the product of the operators of highest orders in the θ subgroups which are separately generated by the $p^{\beta_1}, p^{\beta_2}, \cdots, p^{\beta_\theta}$ powers respectively of the t-generators of the same order contained in G. As the invariants of each of these θ subgroups are equal to each other, these powers for any particular subgroup are evidently I-conjugate, but the powers for one subgroup are not I-conjugate with the powers in question contained in another of these θ subgroups. In particular, the number of operators in each set of I-conjugate operators of G besides identity is divisible by $p-1$, as results also directly from the fact that an automorphism of an abelian group can be obtained by letting each operator correspond to any given power of itself whose index is prime to the order of the group.

If the order of an abelian group is not a power of a prime number, the number of its sets of I-conjugate operators is evidently the product of the numbers of the sets of I-conjugate operators of its Sylow subgroups. In particular, it may be desirable to emphasize the theorem: *The number of sets of I-conjugate operators in any abelian group is equal to the product of the numbers of the I-reduced operators in its Sylow subgroups for a set of independent generators in which the order of each generator is a power of a prime number.* In this theorem, identity is included among the I-reduced operators of a Sylow subgroup.

For the purpose of illustrating the preceding developments, we shall consider the special abelian group of order p^{10} and of type (6, 3, 1). In addition to identity, the number of I-reduced operators involving a single constituent is 10, the number of those involving two constituents is 11, and the number involving three constituents is 2. Hence this group involves 24 sets of I-conjugate operators including identity. The numbers of I-conjugate operators in these 24 sets are as follows: $1, p-1, p^2-p, p^3-p^2, p^5-p^4, p^7-p^6, p^{10}-p^9, p^2-p, p^4-p^3, p^7-p^6, p^3-p^2, (p^2-p)(p-1), (p^3-p^2)(p-1), (p^3-p^2)(p-1), (p^4-p^3)(p-1), (p^4-p^3)(p-1), (p^5-p^4)(p-1), (p^5-p^4)(p-1), (p^7-p^6)(p-1), (p^7-p^6)(p-1),$

$(p^8-p^7)(p-1)$, $(p^4-p^3)(p-1)$, $(p^4-p^3)(p-1)^2$, $(p^5-p^4)(p-1)^2$. These numbers illustrate the obvious theorem that a necessary and sufficient condition that an operator of an abelian group of order p^m, $p>2$, having no two invariants which are equal to each other, be either a possible independent generator or a power of such a generator is that the number of its I-conjugates be not divisible by $(p-1)^2$.

This theorem is evidently a special case of the theorem that a necessary and sufficient condition that the I-reduced operator of a set of I-conjugate operators involve powers of α operators of a set of independent generators of G is that the number of operators in this set be divisible by $(p-1)^\alpha$ for a general value of p. It should be noted that the number of I-conjugate sets of operators of a group of order p^m depends on the type of this group, but is independent of the value of the prime number p, and that the theorem stated at the close of the preceding paragraph is not affected by the number of independent generators of the same order when G has a general value. For special given values of p, the theorem stated at the opening of the present paragraph is clearly not always valid.

III. Criteria for I-conjugate operators and for certain I-conjugate subgroups

It was noted in the preceding section that there is one and only one I-reduced operator in every complete set of I-conjugate operators of the abelian group G of order p^m, and that the number of constituents in terms of a fixed set of independent generators of G appearing in such an I-reduced operator can be determined from the number of operators involved in the set of I-conjugates to which this I-reduced operator belongs. A necessary and sufficient condition that two operators of G be I-conjugate is that they be I-conjugate with the same I-reduced operators. We proceed to develop another criterion for determining when two operators are I-conjugate.

The cyclic group generated by an I-reduced operator gives rise to a quotient group which is known to be simply isomorphic with a subgroup of G. The s-generators of G and all the t-generators of G with respect to this I-reduced operator except those whose powers actually appear in it can also be used as independent generators of the quotient group in question. To each of the latter t-generators, except the smallest one, there corresponds a generator of this quotient group whose order exceeds the order of all these t-generators whose order is less than that of the t-generator in question.

The quotient groups which correspond to two I-conjugate cyclic subgroups are evidently of the same type. To prove that, conversely, every two cyclic subgroups which give rise to quotient groups of the same type are I-conjugate, it should be noted that when these cyclic groups are replaced by those generated

by the I-reduced operators in the sets of I-conjugate operators to which their generators belong, their largest constituent groups with respect to the set of independent generators of G in question must be generated by the same power of independent generators of the same order, since, otherwise, in one quotient group the number of independent generators, beginning with the largest, whose orders coincide with those of G would differ from the number of the corresponding independent generators of the other quotient group.

If the generators of the cyclic groups in question involve powers of more than one t-generator of G, the second t-generator involved must again be the same for both of these cyclic groups, since the independent generators of the quotient group which corresponds to the first t-generator are of a larger order than the second t-generator, as was noted above. Moreover, the same power of this second t-generator must appear in a generator of each of the two cyclic subgroups in question. As this process may be continued until all the t-generators whose powers appear in the constituents of the cyclic subgroups under consideration have been exhausted, there results the following:

THEOREM. *A necessary and sufficient condition that two operators of any abelian group be I-conjugate is that the cyclic groups generated by these operators give rise to quotient groups which are of the same type.*

It results directly from the preceding theorem that the number of different sets of I-conjugate operators can be determined by counting the number of different types of quotient groups to which cyclic subgroups of G give rise. For instance, the cyclic subgroups of the abelian group of order p^4 and of type $(3, 1)$ clearly give rise to quotient groups of the following types, and of no other types: $(3, 1)$, (3), $(2, 1)$, (2), $(1, 1)$, (1). Hence this group has exactly six sets of I-conjugate operators, including identity. The number of operators in these sets is 1, $p^2 - p$, $p - 1$, $p(p-1)^2$, $p^2 - p$, $p^4 - p^3$, respectively. All of these operators are either possible independent generators or powers of such generators except those of the fourth set.

As a first step in a proof of the theorem that if two subgroups of the same type give rise to cyclic quotient groups they must be I-conjugate, it will be convenient to consider a necessary and sufficient condition that a subgroup H of G give rise to a cyclic quotient group. If G/H is cyclic, and if as many as possible of the operators of a set of independent generators of G are selected from the operators of H, the remaining operators of this set can be chosen as follows:

As one of the operators any operator s_1 of lowest order contained in a co-set corresponding to any operator of highest order in G/H may be selected. A necessary and sufficient condition that s_1 be the only operator of the set of independent generators in question which does not appear in H is that one of the operators of smallest order in every co-set corresponding to an operator

of G/H be a power of s_1. When this condition is not satisfied, find one of the largest operators of G/H such that a power of s_1 is not one of the smallest operators in the corresponding co-set. Let s_2 be any one of the smallest operators in such a co-set. It is evident that s_2 may then be chosen as a second operator of the set of independent generators in question.

If the powers of s_2 are not operators of lowest order in the co-sets with respect to H in which they appear, we select an operator s_3 of lowest order in the co-set which corresponds to the largest operator of G/H to which such an operator corresponds but is not an operator of lowest order in the co-set. This process is continued until an operator s_α is found such that its powers are operators of lowest order in all the co-sets with respect to H in which they appear. The operators $s_1, s_2, \cdots, s_\alpha$ are then the operators of the set of independent generators in question which do not appear in H. It may be noted that the ratio of the order of any one of these operators and the order of the one which follows it in this sub-set cannot be less than p^2.

For the independent generators of H which are not also independent generators of G we may choose the operators

$$s_1^{p^{\rho_1}} s_2, s_2^{p^{\rho_2}} s_3, \cdots, s_{\alpha-1}^{p^{\rho_{\alpha-1}}} s_\alpha, s_\alpha^{\rho_\alpha},$$

where $s_x^{p^{\rho_x}}$ $(x = 1, 2, \cdots, \alpha - 1)$ is the inverse of the lowest power of s_x which appears in a co-set with respect to H in which it is not an operator of lowest order, and $s_\alpha^{\rho_\alpha}$ is the lowest power of s_α which is found in H. The order of $s_x^{p^{\rho_x}}$ must exceed the order of s_{x+1}, since the latter is an operator of lowest order in the co-set in which the former appears and is an independent generator which cannot be replaced by an operator of H. Hence it follows that *a necessary and sufficient condition that a subgroup G give rise to a cyclic quotient group is that the* α *independent generators* $(s_1, s_2, \cdots, s_\alpha)$ *of G which cannot be selected from H can be so chosen that*

$$s_\alpha^{p^{\rho_\alpha}} \quad \textit{and} \quad s_x^{p^{\rho_x}} s_{x+1} \qquad (x = 1, 2, \cdots, \alpha - 1)$$

are independent generators of H, where $s_x^{p^{\rho_x}}$ *is of a larger order than* s_{x+1} *and* $\rho_x > 0$. It should be noted that each of these independent generators of H is of a larger order than any of the independent generators of G whose powers appear in the succeeding independent generators of H.

By means of the theorem of the preceding paragraph, it is easy to find a necessary and sufficient condition that two subgroups H_1, H_2 of G which give rise to cyclic quotient groups be I-conjugate. It is evident that a necessary condition is that H_1 and H_2 be of the same type. To prove that this is also a sufficient condition, when it is assumed that both of the quotient groups G/H_1 and G/H_2 are cyclic, let $s_1, s_2, \cdots, s_\rho$ and $t_1, t_2, \cdots, t_\rho$ be two sets of independent generators of G which have been so chosen that as many as possible

of these operators are selected from those of H_1 and H_2 respectively. It results that the first operators of each of these sets, arranged in the descending order of magnitude, whose orders exceed the order of the corresponding reduced independent generator of H_1 and H_2, respectively, arranged similarly, must have the same order. The largest independent generator of H_1 which is not also an independent generator of G has the same order as the largest independent generator of H_2 which is not also an independent generator of G, since the order of this independent generator must exceed the order of all the other independent generators of G which are not also independent generators of H_1 or H_2. Hence it results that these independent generators of H_1 and H_2 may be regarded as products of powers of independent generators of G which are of the same order and independent generators of next to the highest order which are found in G but not in the H's.

Just as the operators of H_1 and H_2 which correspond to the largest independent generator of G which is not also an independent generator of H_1 or H_2 can be chosen from the operators of G as $s_1^{p^{\rho_1}} s_2$ was chosen, so the operators which correspond to the next to the largest independent generator of G which is not also an independent generator of H_1 or H_2 can be chosen in the same way as $s_2^{p^{\rho_2}} s_3$ was chosen, whenever not all the independent generators of G save one can be selected from the operators of H_1. Since these arguments apply to these successive independent generators, it results that *every two subgroups of G which are both of the same type and give rise to cyclic quotient groups are I-conjugate.*

If H_1 and H_2 are two subgroups of the same type which give rise to two quotient groups of the same type, it does not necessarily follow that H_1 and H_2 are *I*-conjugate, as may be seen by considering the group G of order p^9 and of type (5, 3, 1). If s_1, s_2, s_3 represent the three generators of G of orders p^5, p^3, and p respectively, and if $s_1^p s_2$, s_2^p and s_1^p, $s_2^p s_3$ are the independent generators of H_1 and H_2 respectively, it results that the two quotient groups G/H_1 and G/H_2 are of type (2, 1), and the two groups H_1, H_2 are of type (4, 2). The latter groups cannot be *I*-conjugate, since the operators of the highest order in the latter are powers of operators of highest order in G, but this is not the case as regards the operators of highest order of the former subgroup.

It may be noted that the sum of the number of independent generators of a subgroup of G plus the number of the independent generators of the quotient group corresponding to this subgroup is equal to the number of independent generators of G whose common order is p increased by a number which may vary from the number of independent generators of G whose orders exceed p to twice this number, but can have no other value. Both of the limiting values can evidently be actually attained, and the fact that this sum can have no other values results from the theorem that a quotient group of an abelian

group is always simply isomorphic with a subgroup of this group, and that the independent generators of a subgroup which gives rise to a cyclic quotient group can be selected as noted above.

It was noted above that when two subgroups of the same type give rise to cyclic quotient groups they must be I-conjugate, and when two cyclic subgroups give rise to quotient groups of the same type they are also I-conjugate. The other extreme cases are when two subgroups of the same type give rise to quotient groups of type $(1, 1, 1, \cdots)$ and when two subgroups of type $(1, 1, 1, \cdots)$ give rise to quotient groups of the same type. In each of these two cases the two subgroups in question are again I-conjugate. In the special case when a subgroup gives rise to a quotient group of type $(1, 1, 1, \cdots)$ which involves as many invariants as G itself, the subgroup is characteristic, being the ϕ-subgroup of G. In this special case, the subgroup is completely determined by the type of the quotient group to which it gives rise.

Every subgroup of G which gives rise to a quotient group of type $(1, 1, 1, \cdots)$ must include the ϕ-subgroup of G. If the ϕ-quotient group is of order p^α and a subgroup H gives rise to a quotient group of order p^β and of type $(1, 1, 1, \cdots)$, it results that exactly $\alpha - \beta$ of the independent generators of G are found in H, while the pth power of each of the other independent generators of G is found in this subgroup. From this it results directly that if two subgroups of the same type give rise to quotient groups of type $(1, 1, 1, \cdots)$ these subgroups must be I-conjugate. The number of the characteristic subgroups which give rise to quotient groups of type $(1, 1, 1, \cdots)$ is evidently equal to the number of the different orders of reduced independent generators of G. As every operator of order p found in G is a power of a possible independent generator of G, it results that when two subgroups of type $(1, 1, 1, \cdots)$ give rise to quotient groups of the same type their independent generators can be so chosen that they are powers of independent generators of G which are of the same orders. Hence these subgroups are I-conjugate.

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